Equivariant PDEs and the freezing method

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Overview

- Equivariant evolution equations
- The freezing method
- nonlinear waves in 1D,2D,and 3D
- Relative equilibria and relative periodic orbits
- Asymptotic stability and freezing
- Decomposition of multifronts and multipulses
- Summary and perspectives

Equivariant evolution equations

$$\boxed{\begin{array}{c} u_t = F(u), \quad u(0) = u_0, \\ F : Y \subset X \to X, \quad X \text{ Banach space, } Y \text{ dense} \end{array}} \qquad \text{(EV)}$$

G a (noncompact) Lie group acting on X via

$$\boxed{a : G \times X \to X, \ (\gamma, v) \mapsto a(\gamma)v}$$

$$\boxed{a(\gamma) \in \mathrm{GL}(X), a(\mathbb{1}) = I, a(\gamma_1 \circ \gamma_2) = a(\gamma_1)a(\gamma_2)} \quad \text{(Homomorph.)}$$

$$\boxed{\begin{array}{c} F(a(\gamma)u) = a(\gamma)F(u) \quad \forall u \in Y, \gamma \in G \\ a(\gamma)(Y) \subset Y \quad \forall \gamma \in G \end{array}} \qquad \text{(Equivariance)}$$

$$\boxed{\begin{array}{c} a(\cdot)v : \ \gamma \mapsto a(\gamma)v \text{ continuous } \forall v \in X, \\ \text{differentiable } \forall v \in Y \text{ with derivative } \ d[a(\gamma)v] \end{array}} \qquad \text{(Smoothness)}$$

Parabolic system

$$\boxed{u_t = Au_{xx} + f(u, u_x), \qquad x \in \mathbb{R},\ u(x,t) \in \mathbb{R}^m} \qquad \text{(PDE)}$$

Settings: $X = \mathcal{L}_2(\mathbb{R}), Y = H^2(\mathbb{R}), \mathbb{R} = G, \mathbb{1} = 0$

$$\begin{aligned} [a(\gamma)v](x) &= \quad v(x-\gamma), \quad v \in X, \gamma \in G, \\ d[a(\gamma)v] &= \ -v_x(\cdot - \gamma), \quad v \in Y, \gamma \in G, \\ F(v) &= \quad v_{xx} + f(v, v_x), \quad v \in Y. \end{aligned}$$

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Topics:

- Traveling waves (relative equilibria) come in families.
- Use equivariance for solving the Cauchy problem $u(x,0) = u_0(x)$.
- Relate asymptotic stability to spectrum of linearization.
- Use equivariance for bifurcation analysis.

The freezing method: Write the solution $u(x,t)$ of

$$\boxed{u_t = Au_{xx} + f(u, u_x), \qquad u(x,0) = u_0(x), x \in \mathbb{R}} \quad \text{(CAUCHY)}$$

in terms of new unknowns $\gamma(t) \in \mathbb{R}$ and $v(\cdot, t)$ as follows

$$\boxed{u(x,t) = v(x - \gamma(t), t), \quad x \in \mathbb{R}, t \geq 0.}$$

Then solve a Partial Differential Algebraic Equation for $v(\cdot, t), \gamma(t), \mu(t)$

$$\boxed{\begin{aligned} v_t &= Av_{xx} + f(v, v_x) + \mu v_x, \quad v(\cdot, 0) = u_0 \\ 0 &= \langle \hat{v}_x, v(\cdot, t) - \hat{v} \rangle_{\mathcal{L}_2} \quad \text{phase condition} \\ \gamma_t &= \mu(t), \quad \gamma(0) = 0 \end{aligned}} \quad \text{(PDAE)}$$

- $\hat{v}$ a template function, details below,
- traveling wave $u(x,t) = \bar{v}(x - ct)$ is an equilibrium of (PDAE),
- The systems (PDAE) and (CAUCHY) are equivalent on $\mathbb{R}$, but have different longtime behavior on $[x_-, x_+]$.

Phase conditions

- Fixed phase condition:
  Require $\|\hat{v}(\cdot-\gamma)-v(\cdot,t)\|_{\mathcal{L}_2}$ to be minimal at $\gamma=0$ for some template function $\hat{v}$. Leads to $0=\langle \hat{v}_x, v(\cdot,t)-\hat{v}\rangle$.
  DAE of index 2: Differentiate w.r.t. t and obtain

$$\boxed{0=\langle \hat{v}_x, v_t\rangle = \mu\langle \hat{v}_x, v_x\rangle + \langle \hat{v}_x, v_{xx}+f(v,v_x)\rangle =: \psi_{\text{fix}}(v,\mu)}$$

- Orthogonality condition
  Require $\|v_t(\cdot,t)\|_{\mathcal{L}_2}$ to be minimal at each t, so necessarily $0=\frac{d}{d\mu}||Av_{xx}+f(v,v_x)+\mu v_x||^2_{\mathcal{L}_2}$ at $\mu=\mu(t)$,
  leads to a DAE of index 1

$$\boxed{0=\mu\langle v_x, v_x\rangle + \langle v_x, v_{xx}+f(v,v_x)\rangle =: \psi_{\text{orth}}(v,\mu)}$$

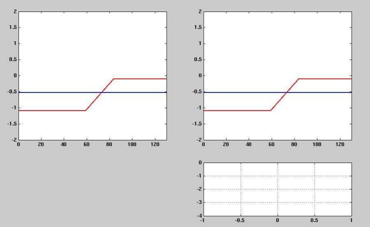
FitzHugh-Nagumo wave

$$\boxed{\begin{aligned} V_t &= \Delta V + V - \frac{1}{3}V^3 - R + \mu V_x, \\ R_t &= \phi(V + a - bR) + \mu R_x \end{aligned}}$$

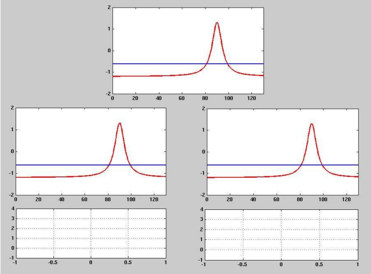
$J = [0, 130]$, $\Delta x = 0.5$, $\Delta t = 0.01$,$a = 0.7$,$b = 0.8$, $\phi = 0.08$.

Upwind/downwind for convective term μv_x

$$v_x \approx D_\pm v = \alpha D_+ v + (1-\alpha) D_- v, \quad \alpha = (1 + \mathrm{e}^{-\beta\lambda})^{-1}, \; \beta = 1$$



traveling vs. frozen



ψ_{orth} vs. ψ_{fix} phase condition

Generalization to equivariant equations $u_t = F(u)$

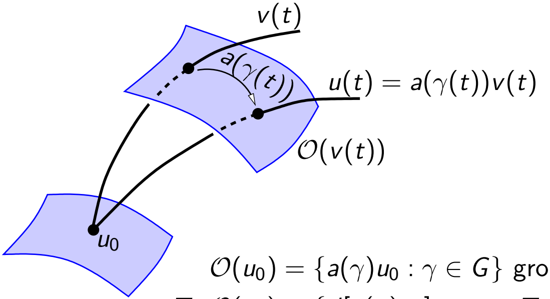
Introduce $\gamma(t) \in G$, $v(t) \in Y$ and write

$$\boxed{u(t) = a(\gamma(t))v(t)}$$

Add a phase condition on v

$$\boxed{\psi(v) = 0, \quad \text{where } \psi : Y \mapsto \mathcal{A}^*}$$

$\mathcal{A}^* = L[\mathcal{A}, \mathbb{R}]$ dual of $\mathcal{A} = T_{\mathbb{1}} G$ (Lie algebra)



$\mathcal{O}(u_0) = \{a(\gamma)u_0 : \gamma \in G\}$ group orbit

$T_{u_0}\mathcal{O}(u_0) = \{d[a(\gamma)u_0]\mu : \mu \in T_\gamma G\}$ tangent space

Resulting equation

Insert $u(t) = a(\gamma(t))v(t)$ into (EV)

$$a(\gamma)F(v) = F(a(\gamma)v) = \boxed{F(u) = u_t} = a(\gamma)v_t + \ d[a(\gamma)v]\gamma_t$$

Introduce $\mu(t) = \mu \in \mathcal{A} = T_{\mathbb{1}}G$ (Lie algebra) via $\gamma_t = \ dL_\gamma(\mathbb{1})\mu$, where dL_γ is the derivative of $L_\gamma : G \to G,\ g \mapsto \gamma \circ g$.
Solve for $\gamma(t) \in G, \mu(t) \in \mathcal{A}, v(t) \in Y$

$$\boxed{\begin{aligned} v_t &= F(v) - \ d[a(\mathbb{1})v]\mu, && v(0) = u_0 \\ \gamma_t &= \ dL_\gamma(\mathbb{1})\mu, && \gamma(0) = \mathbb{1} \\ 0 &= \psi(v, \mu) \end{aligned}} \qquad \text{(DAEV)}$$

- related approach: Rowley, Kevrekidis, Marsden and Lust 2003
- The term $d[a(\mathbb{1})v]\mu$ in (DAEV)is obtained by applying $\frac{d}{dg}$ to $a(\gamma)a(g)v = a(L_\gamma g)v$ at $g = \mathbb{1}$

$$\boxed{a(\gamma)d[a(\mathbb{1})v]\mu = d[a(\gamma)v]dL_\gamma(\mathbb{1})\mu \quad \text{for all } \mu \in \mathcal{A}.}$$

Reaction diffusion systems in $\mathbb{R}^2$

$$\boxed{\begin{aligned} u_t &= \Delta u + f(u), \quad t \geq 0 \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^2 \end{aligned}}$$

Action of Euclidean group

$$\boxed{\begin{gathered} G = SE(2) = S^1 \ltimes \mathbb{R}^2 \ \ni \ \gamma = (\phi, \tau) \\ [a(\gamma)v](x) = v(R_{-\phi}(x - \tau)) \end{gathered}}$$

with the rotations

$$R_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

group operation

$$(\phi_1, \tau_1) \circ (\phi_2, \tau_2) = (\phi_1 + \phi_2, \tau_1 + R_{\phi_1}\tau_2)$$

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symmetry terms (angular and translational velocities)

$$d[a(\mathbb{1})v]\mu = \mu_1(yv_x - xv_y) + \mu_2 v_x + \mu_3 v_y$$

Quintic Ginzburg Landau equation, 2d

$$u_t = \alpha \Delta u + \delta u + \beta |u|^2 u + \gamma |u|^4 u, \quad (x,y) \in \mathbb{R}^2,\ u(x,y,t) \in \mathbb{C}$$

$\alpha = 0.5(1+i),\ \delta = -0.5,\ \beta = 2.5 + i,\ \gamma = -1 - 0.1i,$
$a(\gamma)v(\xi) = e^{i\theta} v(R_{-\phi}(\xi - \tau))$ for $\gamma = (\phi, \tau, \theta) \in G = SE(2) \times S^1,$

Quintic Ginzburg Landau equation, 2d

$$
\begin{aligned}
u_t &= \alpha\Delta u + \delta u + \beta|u|^2 u + \gamma|u|^4 u \\
&\quad + \mu_1(yu_x - xu_y) + \mu_2 u_x + \mu_3 u_y + \mu_4 iu \\
0 &= \langle yu_{0,x} - xu_{0,y}, u - u_0\rangle_{\mathcal{L}_2}, \quad 0 = \langle iu_0, u - u_0\rangle_{\mathcal{L}_2} \\
0 &= \langle u_{0,x}, u - u_0\rangle_{\mathcal{L}_2}, \quad 0 = \langle u_{0,y}, u - u_0\rangle_{\mathcal{L}_2}
\end{aligned}
$$

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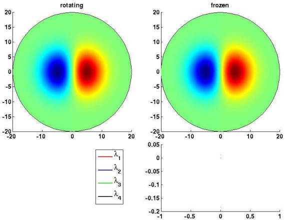
$a(\gamma)v(\xi) = e^{i\theta}v(R_{-\phi}(\xi - \tau))$ for $\gamma = (\phi, \tau, \theta) \in G = SE(2) \times S^1$,
Computation with FEM package COMSOL Multiphysics,
Neumann b.c.



Scroll waves in $\mathbb{R}^3$: CGL-system

$$u_t = \Delta u + (1 - |u|^2 - i|u|^2)u, \quad x \in \mathbb{R}^3,\ u(x,t) \in \mathbb{C}$$

Action of Euclidean group

$$G = SE(3) = SO(3) \ltimes \mathbb{R}^3,\ \gamma = (R, \tau)$$
$$[a(\gamma)v](x) = v(R^{-1}(x - \tau))$$

group operation

$$\gamma \circ \tilde{\gamma} = (R\tilde{R}, \tau + R\tilde{\tau})$$

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$$[a(\gamma)v](x) = v(R^{-1}(x - \tau))$$

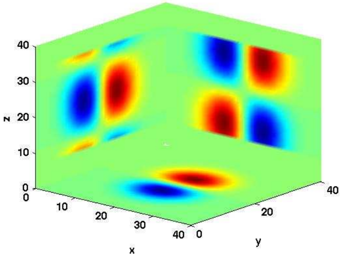
group operation $\gamma \circ \tilde{\gamma} = (R\tilde{R}, \tau + R\tilde{\tau})$

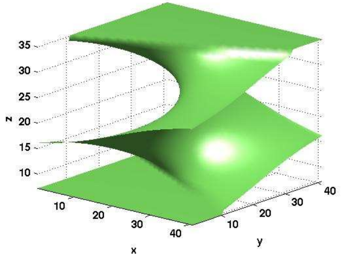
$$v_t = \Delta v + (1 - |v|^2 - i|v|^2)v + \mu_4 v_{x_1} + \mu_5 v_{x_2} + \mu_6 v_{x_3}$$
$$+ \mu_1(v_{x_2}x_3 - v_{x_3}x_2) + \mu_2(v_{x_3}x_1 - v_{x_1}x_3) + \mu_3(v_{x_1}x_2 - v_{x_2}x_1)$$

corresponding phase conditions

Numerical solution with adaptation of `ezscroll` (Barkley '97)
$L_{x_i} = 40$, $\Delta x_i = 1$, $\Delta t = \frac{3}{8}10^{-3}$, 19-point Laplacian,
boundary conditions: x, y - Neumann, z - periodic
initial function $u_0(r, \varphi, z) = \exp(\frac{iz}{2\pi})\frac{r}{40}(\cos(\varphi) + i\sin(\varphi))$

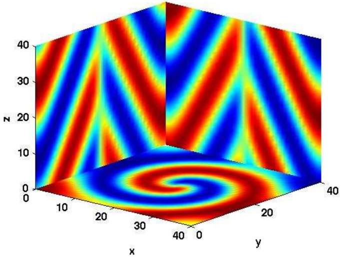
Scroll wave in 3d

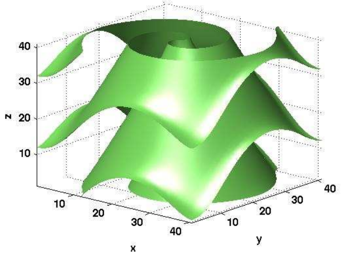




initial cond., x,y,x-slices through origin

isosurface, Re(u)=0





solution at $t = 300$, x,y,x-slices through origin

isosurface, Re(u)=0

Relative equilibria and relative periodic orbits

Definition: A solution $u(t) = a(\gamma(t))\bar{v}, \bar{v} \in Y, \gamma \in C^1(\mathbb{R}_+, G)$ of $u_t = F(u)$ is called a relative equilibrium.
Likewise, a solution $u(t) = a(\gamma(t))\bar{v}(t)$ is called a relative periodic orbit if $\bar{v}(t)$ has nontrivial period $T > 0$.

Characterization: Assume $d[a(\mathbb{1})\bar{v}] : \mathcal{A} \mapsto X$ is one to one. Then $a(\gamma(t))\bar{v}$ is a relative equilibrium iff there exists $\bar{\mu} \in \mathcal{A}$ such that $\bar{v}$ is a steady state of

$$\boxed{v_t = F(v) - d[a(\mathbb{1})v]\bar{\mu}} \qquad \text{(PDAE)}$$

and

$$\boxed{\gamma_t = dL_\gamma(\mathbb{1})\bar{\mu}} \qquad \text{(RE)}$$

Solution of (RE): $\gamma(t) = \exp(t\bar{\mu})\gamma(0)$.

Asymptotic stability and freezing

Goal: Stability of relative equilibrium with asymptotic phase turns into Liapunov stability of the pair $(\bar{v}, \bar{\mu})$.

Stability with asymptotic phase:

A relative equilibrium $a(\gamma(t))\bar{v}$ is called stable with asymptotic phase if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for all solutions of $u_t = F(u)$ with $||u(0) - \bar{v}|| \leq \delta$ there exists $\gamma_0(t) \in G$ satisfying

$$\|u(t) - a(\gamma_0(t))\bar{v}\| \begin{cases} \leq \epsilon, & \forall t \geq 0 \\ \to 0 & \text{for } t \to \infty. \end{cases}$$

(Meta)Theorem: Stability with asymptotic phase holds if and only if $(\bar{v}, \bar{\mu}) \in Y \times \mathcal{A}$ is an asymptotically stable equilibrium of

$$\boxed{\begin{aligned} v_t &= F(v) - d[a(\mathbb{1})v]\mu, & v(0) &= u_0 \\ \gamma_t &= dL_\gamma(\mathbb{1})\mu, & \gamma(0) &= \mathbb{1} \\ 0 &= \psi(v, \mu) \end{aligned}} \qquad \text{(DAEV)}$$

in the Lyapunov sense for all consistent initial data.

Results: Traveling waves in 1D, rotating waves in 2D.

Stability of two-dimensional rotating patterns

with J.Lorenz DPDE, 2009.
For the reaction diffusion system in $\mathbb{R}^2$

$$\boxed{u_t = A\Delta u + f(u), \quad u(x,0) = u_0(x) \quad x \in \mathbb{R}^2,\ t \geq 0} \qquad \text{(RD)}$$

consider a rigidly rotating localized pattern

$$\boxed{u(x,t) = \bar{v}(R_{-ct}x), c > 0, R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \qquad \text{(LP)}$$

- $\sup_{|x|\geq R} |\bar{v}(x) - u_\infty| \to 0 \quad \text{as} \quad R \to \infty$,
- $f(u_\infty) = 0$ and $f'(u_\infty) \leq -\beta I < 0$,
- the eigenvalues $\pm ic$ with eigenvector $D_1\bar{v} \pm iD_2\bar{v}$ and 0 with eigenvector $D_\phi\bar{v}$ are simple for the linearized operator $\mathcal{L} = A\Delta + cD_\phi + f'(\bar{v})$ in $H^2_{Eucl} = \{u \in H^2 : D_\phi u \in L^2\}$,
- $\mathcal{L}$ has no further eigenvalues $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) \geq -\beta$.

Nonlinear stability for rotating localized 2D-patterns

Theorem (with J. Lorenz)

Under the assumptions above there exists an $\varepsilon > 0$ *such that for any solution of* (RD) *satisfying* $||u_0 - \bar{v}||_{\mathcal{H}^2} \leq \varepsilon$ *there is a* C^1*-function* $\gamma(t) = (\theta(t), \tau(t)) \in SE(2)$ *and* $(\theta_\infty, \tau_\infty) \in SE(2)$ *with*

$$
\boxed{\begin{aligned} ||u(\cdot, t) - a(\gamma(t))\bar{v}||_{\mathcal{H}^2} &\leq C \exp(-\frac{\beta}{2} t) ||u_0 - \bar{v}||_{\mathcal{H}^2} \\ |\theta(t) + ct - \theta_\infty| + |\tau(t) - \tau_\infty| &\leq C \exp(-\frac{\beta}{2} t) ||u_0 - \bar{v}||_{\mathcal{H}^2} \end{aligned}}
$$

From the proof:

- $\mathcal{L} = A\Delta + cD_\phi + f'(\bar{v})$ not sectorial, $\sigma_{\text{ess}}(\mathcal{L})$ contains curves, $s = \lambda_j(\kappa) + inc, n \in \mathbb{Z}, \kappa \in \mathbb{R}$, $\lambda_j(\kappa)$ ev. of $-\kappa^2 A + f'(u_\infty)$.
- Compact perturbation theorem for C^0-semigroups applies in $\mathcal{H}^2$ after splitting off the trivial eigenvalues $\pm ic, 0$,
- Use Sobolev and Gagliardo Nirenberg estimates in $\mathcal{H}^2$.

Remark: Convergence of freezing method still to be proved.

Essential spectrum

Linearization in polar coordinates

$$\boxed{\mathcal{L} = A\left(D_r^2 + \frac{1}{r}D_r + \frac{1}{r^2}D_\phi^2\right) + cD_\phi + f'(\bar{v}(r,\phi))}$$

In the far field ($r = \infty$): $\mathcal{L}_{far} = AD_r^2 + cD_\phi + f'(u_\infty)$.
Find solutions of $u_t = \mathcal{L}_{far}u$ that take the form
$u(r,\phi,t) = e^{st}e^{in\phi}e^{i\kappa r}v$, $r \geq 0, \phi \in [0, 2\pi]$ for some $v \in \mathbb{C}^m$,

$$\boxed{\det(-\kappa^2 A + inc + f'(u_\infty) - s) = 0 \quad \text{dispersion relation}}$$

Theorem

If s satisfies the dispersion relation for some $\kappa \in \mathbb{R}$, $n \in \mathbb{Z}$, then $s \in \sigma_{ess}(\mathcal{L})$.

Method of proof: For $u_R(r,\phi) = \chi_R e^{i(n\phi+\kappa r)}v$
(v an eigenvector, χ_R a cut-off function) show

$$\|(\mathcal{L} - s)u_R\|_{\mathcal{L}^2} \leq C, \quad \|u_R\|_{\mathcal{L}^2} \geq C\sqrt{R}$$

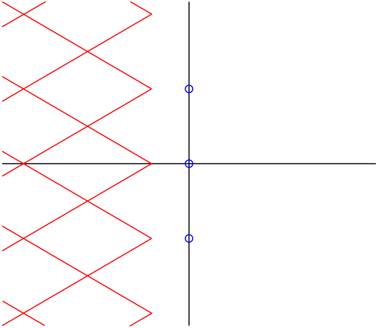
Example: Quintic Ginzburg Landau

$$u_t = \alpha\Delta u + \delta u + \beta|u|^2 u + \gamma|u|^4 u, \quad (x,y) \in \mathbb{R}^2, t \geq 0.$$

Infinitely many copies of two half lines

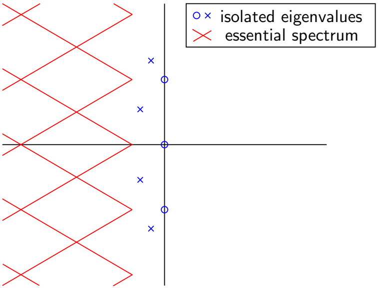
$$s = -\kappa^2\alpha + inc + \delta, \quad s = -\kappa^2\bar{\alpha} + inc + \bar{\delta}, \quad \kappa \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

Data: $\alpha = \frac{1+i}{2}, \delta = -\frac{1}{2} < 0$



essential spectrum: $s = inc + \delta - \kappa^2(\alpha_1 \pm i\alpha_2), \kappa \in \mathbb{R}, n \in \mathbb{Z}$

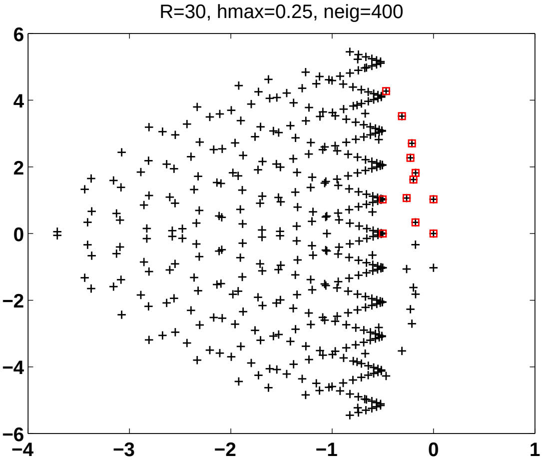
Generic picture of full spectrum



Essential spectrum, critical eigenvalues, and further isolated eigenvalues

Semigroup $e^{t\mathcal{L}}$ is continuous but not analytic !

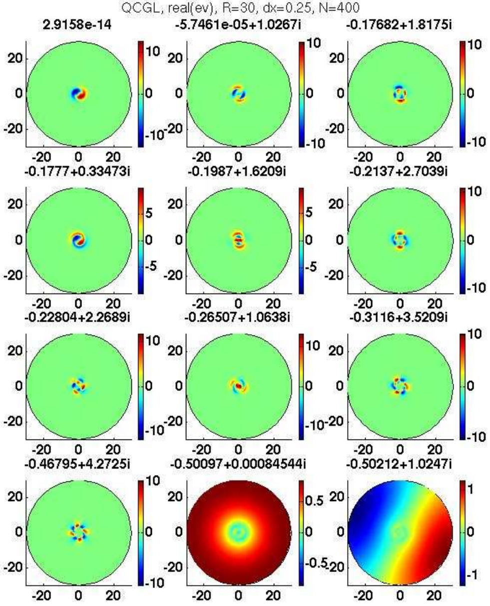
Part of numerical spectrum: 400 ev, system size $\approx 10^5$



8 additional pairs of isolated eigenvalues.

Real parts of eigenfunctions

2 critical and 8 extra isolated eigenvalues, 2 'non-eigenvalues'



Freezing Multifronts and Multipulses

W.-J.B.,Selle,Thümmler, 2008.

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, t \geq 0, u(x,t) \in \mathbb{R}^m$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}$$

Ansatz for decomposition into single fronts

$$u(x,t) = \sum_{j=1}^{N} v_j(x - g_j(t), t), \quad g_j(t) \text{ position of } j\text{-front at time } t$$

Take a bump function $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that
$0 < \varphi(x) \leq 1 \quad \forall x \in \mathbb{R}$
and use a partition of unity depending on positions g_j

$$Q_j(g, x) = \frac{\varphi(x - g_j)}{\sum_{k=1}^{N} \varphi(x - g_k)}, \quad x \in \mathbb{R}, j = 1, \ldots, N.$$

Insert ansatz into the PDE.

Abbreviate $v_k(\cdot) = v_k(\cdot - g_k(t), t)$ and find

$$
\begin{aligned}
u_t &= \sum_{j=1}^{N} [v_{j,t}(\cdot) - v_{j,x}(\cdot) g_{j,t}] \\
&= \sum_{j=1}^{N} \left[v_{j,xx}(\cdot) + Q_j(g,\cdot) f\left(\sum_{k=1}^{N} v_k(\cdot) \right) \right] \\
&= \sum_{j=1}^{N} \left[v_{j,xx} + f(v_j) + Q_j(g,\cdot) \left(f\left(\sum_{k=1}^{N} v_k(\cdot) \right) - \sum_{k=1}^{N} f(v_k(\cdot)) \right) \right].
\end{aligned}
$$

Require that the terms $[\ldots\ldots]$ in the first and in the third sum match.

Set

$$\xi = x - g_j(t), \quad j = 1, \dots, N$$
$$*_{kj} = \xi - g_k(t) + g_j(t) \quad j, k = 1, \dots, N.$$

Then a sufficient condition for the system to be satisfied is

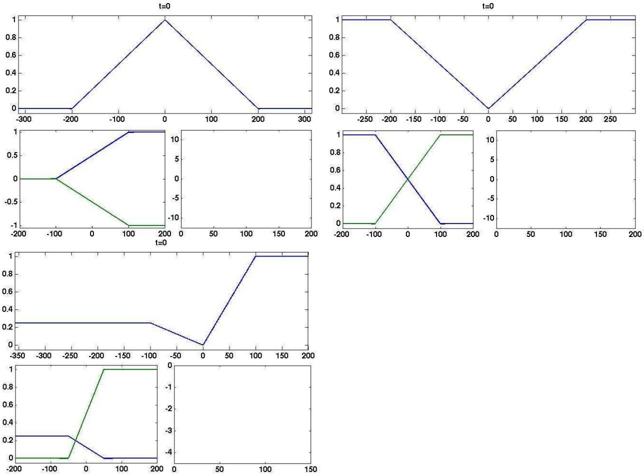
$$\begin{aligned}
v_{j,t}(\xi, t) &= v_{j,\xi\xi}(\xi, t) + v_{j,\xi}(\xi, t)\mu_j(t) + f(v_j(\xi, t)) \\
&\quad + \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(*_{kj})}\left[f\left(\sum_{k=1}^N v_k(*_{kj}, t)\right) - \sum_{k=1}^N f\left(v_k(*_{kj}, t)\right)\right] \\
0 &= \langle v_j(\cdot, t) - \hat{v}_j, \hat{v}_{j,x}\rangle_{\mathcal{L}_2}, \quad v_j(\cdot, 0) = v_j^0(\cdot), \\
g_{j,t} &= \mu_j, \quad g_j(0) = g_{j,0}.
\end{aligned}$$

- PDAE-System with nonlinear and nonlocal coupling.
- Decomposition is generally not unique !
- For fronts one needs a modified version.
- Solve the above system on a finite interval $[x_-, x_+]$.

Nagumo equation

Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-a), \quad x \in \mathbb{R}, t \geq 0, a = \frac{1}{4}$$



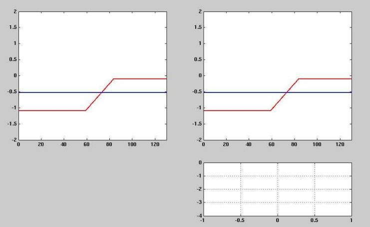
Recall the FitzHugh-Nagumo wave

$$
\boxed{\begin{aligned} V_t &= \Delta V + V - \frac{1}{3}V^3 - R + \mu V_x, \\ R_t &= \phi(V + a - bR) + \mu R_x \end{aligned}}
$$

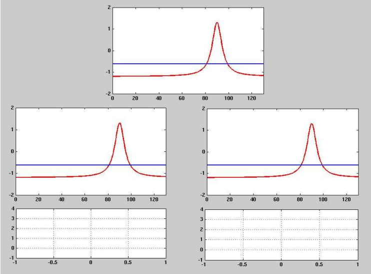
$J = [0, 130]$, $\Delta x = 0.5$, $\Delta t = 0.01$, $a = 0.7$, $b = 0.8$, $\phi = 0.08$.

Upwind/downwind for convective term μv_x

$$
v_x \approx D_\pm v = \alpha D_+ v + (1-\alpha) D_- v, \quad \alpha = (1 + \mathrm{e}^{-\beta\mu})^{-1}, \; \beta = 1
$$

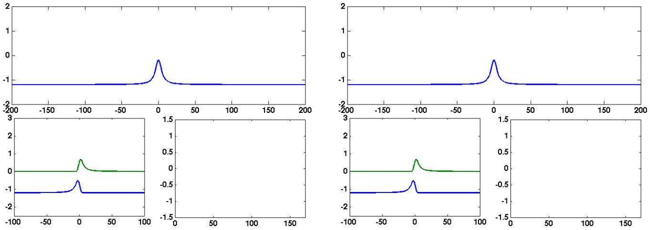


traveling vs. frozen



ψ_{orth} vs. ψ_{fix} phase condition

Decompose and freeze



Theory (`S. Selle 2009`)
Asymptotic stability with asymptotic phase for the coupled system with nonlocal terms in case of weak interaction, i.e. initial data are close to a superposition of waves that are far apart and that are asymptotically stable individually.
Related work on the original equation by `D.Wright 2008/09`.

Summary

- Freezing allows to adaptively compute moving coordinate systems for equivariant PDEs
- Leads to (P)DAEs of index 1 or 2 with additional convective terms
- For parabolic systems in 1D the effects of the transformation

  PDE $\overset{\text{freezing}}{\longrightarrow}$ PDAE $\overset{\text{discretization}}{\longrightarrow}$ DAE have been analyzed
- Stability with asymptotic phase is converted into Lyapunov stability (proved for traveling waves in parabolic and certain hyperbolic systems, almost proved for 2D-rotating patterns in parabolic systems).
- Works for numerical examples in 1D-3D. Artificial convection can create problems for discretizations.
- Multifronts and multipulses with different speeds can be frozen independently (`B.,Selle,Thümmler 2008`), nonlinear stability theorem for the case of weak interaction (`Selle 2009`).

Perspectives

- freezing of relative equilibria for different types of equations
  - viscous conservation laws,
  - Nonlocal diffusion terms,
  - SPDEs (`G.Lord, V.Thümmler`)
- More general equivariance $b(\gamma)F(u) = F(a(\gamma)u)$ occurs when rescaling time, see `Rowley et al.2003`
- Stability proofs for the PDAE formulation in dimensions ≥ 2
- Systems with relative periodic orbits and their direct computation (`Wulff,Schebesch 2006`, `Champneys, Sandstede 2007`)
- Extension of the 'decompose and freeze' approach to multistructures in dimensions ≥ 2.

References

W.-J. B., V. Thümmler, *Freezing solutions of equivariant evolution equations*, SIADS, 2004.

W.-J. B., J. Lorenz, *Nonlinear stability of rotating patterns*, Dynamics of PDEs, 2009.

V. Thümmler, *Numerical approximation of relative equilibria for equivariant PDEs*, SIAM J. Numer. Anal. 2008.

V. Thümmler, *Asymptotic stability of discretized and frozen relative equilibria*, J Dynam. Diff. Eq. 2008.

W.-J. B., V. Thümmler, *Phase conditions, symmetries, and PDE continuation* in Springer Series in Complex Systems (B. Krauskopf, H. Osinga Eds.) 2007.

W.-J. B., S. Selle, V. Thümmler, *Freezing multipulses and multifronts*, SIADS, 2008.

S. Selle, *Decomposition and stability of multipulses and multifronts*, (PhD Thesis), 2009.

J. Rottmann-Matthes, *Computation and stability of patterns in hyperbolic-parabolic systems*, (PhD Thesis), 2010.