Equivariant PDEs
and the freezing method

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With Vera Thümmler, Sabrina Selle (Bielefeld)
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Overview

- Equivariant evolution equations
- The freezing method
- Nonlinear waves in 1D, 2D, and 3D
- Relative equilibria and relative periodic orbits
- Asymptotic stability and freezing
- Decomposition of multifronts and multipulses
- Summary and perspectives
Equivariant evolution equations

$$u_t = F(u), \quad u(0) = u_0,$$

$$F : Y \subset X \rightarrow X, \quad X \text{ Banach space, } Y \text{ dense}$$

$G$ a (noncompact) Lie group acting on $X$ via

$$a : G \times X \rightarrow X, \ (\gamma, v) \mapsto a(\gamma)v$$

$$a(\gamma) \in \text{GL}(X), \ a(1) = I, \ a(\gamma_1 \circ \gamma_2) = a(\gamma_1)a(\gamma_2)$$

(Equivariance)

$$F(a(\gamma)u) = a(\gamma)F(u) \quad \forall u \in Y, \gamma \in G$$

$$(\text{Equivariance})$$

$$a(\gamma)(Y) \subset Y \quad \forall \gamma \in G$$

$$a(\cdot)v : \gamma \mapsto a(\gamma)v \text{ continuous } \forall v \in X,$$
$$\text{differentiable } \forall v \in Y \text{ with derivative } d[a(\gamma)v]$$

(Smoothness)
Parabolic system

\[ u_t = Au_{xx} + f(u, u_x), \quad x \in \mathbb{R}, \; u(x, t) \in \mathbb{R}^m \]  

(PDE)

Settings: \( X = L_2(\mathbb{R}), \; Y = H^2(\mathbb{R}), \; \mathbb{R} = G, \; 1 = 0 \)

\[ [a(\gamma)v](x) = v(x - \gamma), \; v \in X, \gamma \in G, \]

\[ d[a(\gamma)v] = -v_x(\cdot - \gamma), \; v \in Y, \gamma \in G, \]

\[ F(v) = v_{xx} + f(v, v_x), \; v \in Y. \]
Parabolic system

\[ u_t = Au_{xx} + f(u, u_x), \quad x \in \mathbb{R}, \ u(x, t) \in \mathbb{R}^m \]  

(PDE)

Settings: \( X = L_2(\mathbb{R}), \ Y = H^2(\mathbb{R}), \ \mathbb{R} = G, \ \mathbb{1} = 0 \)

\[ [a(\gamma)v](x) = v(x - \gamma), \quad v \in X, \ \gamma \in G, \]
\[ d[a(\gamma)v] = -v_x(\cdot - \gamma), \quad v \in Y, \ \gamma \in G, \]
\[ F(v) = v_{xx} + f(v, v_x), \quad v \in Y. \]

Topics:

- Traveling waves (relative equilibria) come in families.
- Use equivariance for solving the Cauchy problem \( u(x, 0) = u_0(x) \).
- Relate asymptotic stability to spectrum of linearization.
- Use equivariance for bifurcation analysis.
The freezing method: Write the solution \( u(x, t) \) of

\[
\begin{align*}
  u_t &= A u_{xx} + f(u, u_x), \\
  u(x, 0) &= u_0(x), \quad x \in \mathbb{R}
\end{align*}
\]

(CAUCHY)

in terms of new unknowns \( \gamma(t) \in \mathbb{R} \) and \( v(\cdot, t) \) as follows

\[
  u(x, t) = v(x - \gamma(t), t), \quad x \in \mathbb{R}, \quad t \geq 0.
\]

Then solve a Partial Differential Algebraic Equation for \( v(\cdot, t), \gamma(t), \mu(t) \)

\[
\begin{align*}
  v_t &= A v_{xx} + f(v, v_x) + \mu v_x, \quad v(\cdot, 0) = u_0 \\
  0 &= \langle \hat{v}_x, v(\cdot, t) - \hat{v} \rangle_{L^2} \quad \text{phase condition} \\
  \gamma_t &= \mu(t), \quad \gamma(0) = 0
\end{align*}
\]

(PDAE)

- \( \hat{v} \) a template function, details below,
- traveling wave \( u(x, t) = \bar{v}(x - ct) \) is an equilibrium of (PDAE),
- The systems (PDAE) and (CAUCHY) are equivalent on \( \mathbb{R} \), but have different longtime behavior on \([x_-, x_+].\)
Phase conditions

- **Fixed phase condition:**
  Require $\|\hat{\mathbf{v}}(\cdot - \gamma) - \mathbf{v}(\cdot, t)\|_{L^2}$ to be minimal at $\gamma = 0$ for some template function $\hat{\mathbf{v}}$. Leads to $0 = \langle \hat{\mathbf{v}}_x, \mathbf{v}(\cdot, t) - \hat{\mathbf{v}} \rangle$.
  DAE of index 2: Differentiate w.r.t. $t$ and obtain
  
  $0 = \langle \hat{\mathbf{v}}_x, \mathbf{v}_t \rangle = \mu \langle \hat{\mathbf{v}}_x, \mathbf{v}_x \rangle + \langle \hat{\mathbf{v}}_x, \mathbf{v}_{xx} + f(\mathbf{v}, \mathbf{v}_x) \rangle =: \psi_{\text{fix}}(\mathbf{v}, \mu)$

- **Orthogonality condition**
  Require $\|\mathbf{v}_t(\cdot, t)\|_{L^2}$ to be minimal at each $t$, so necessarily
  
  $0 = \frac{d}{d\mu}\|A\mathbf{v}_{xx} + f(\mathbf{v}, \mathbf{v}_x) + \mu \mathbf{v}_x\|_{L^2}^2$ at $\mu = \mu(t)$,
  leads to a DAE of index 1
  
  $0 = \mu \langle \mathbf{v}_x, \mathbf{v}_x \rangle + \langle \mathbf{v}_x, \mathbf{v}_{xx} + f(\mathbf{v}, \mathbf{v}_x) \rangle =: \psi_{\text{orth}}(\mathbf{v}, \mu)$
FitzHugh-Nagumo wave

\[
\begin{align*}
V_t &= \Delta V + V - \frac{1}{3} V^3 - R + \mu V_x, \\
R_t &= \phi(V + a - bR) + \mu R_x
\end{align*}
\]

\[J = [0, 130], \ \Delta x = 0.5, \ \Delta t = 0.01, a = 0.7, b = 0.8, \ \phi = 0.08.\]

Upwind/downwind for convective term \(\mu v_x\)

\[v_x \approx D_\pm v = \alpha D_+ v + (1 - \alpha) D_- v, \quad \alpha = (1 + e^{-\beta\lambda})^{-1}, \ \beta = 1\]

traveling vs. frozen

\(\psi_{\text{orth}}\) vs. \(\psi_{\text{fix}}\) phase condition
Generalization to equivariant equations $u_t = F(u)$

Introduce $\gamma(t) \in G$, $\nu(t) \in Y$ and write

$$u(t) = a(\gamma(t))\nu(t)$$

Add a phase condition on $\nu$

$$\psi(\nu) = 0, \quad \text{where } \psi : Y \to \mathcal{A}^*$$

$\mathcal{A}^* = L[\mathcal{A}, \mathbb{R}]$ dual of $\mathcal{A} = T_1 G$ (Lie algebra)

$\mathcal{O}(\nu(t))$

$\mathcal{O}(u_0) = \{ a(\gamma)u_0 : \gamma \in G \}$ group orbit

$T_{u_0}\mathcal{O}(u_0) = \{ d[a(\gamma)u_0]\mu : \mu \in T_\gamma G \}$ tangent space
Resulting equation

Insert \( u(t) = a(\gamma(t))v(t) \) into (EV)

\[
a(\gamma)F(v) = F(a(\gamma)v) = \boxed{F(u) = u_t} = a(\gamma)v_t + d[a(\gamma)v]_{\gamma_t}
\]

Introduce \( \mu(t) = \mu \in A = T_1G \) (Lie algebra) via \( \gamma_t = dL_{\gamma}(\mathbb{1})\mu \), where \( dL_{\gamma} \) is the derivative of \( L_{\gamma} : G \to G, \ g \mapsto \gamma \circ g \).

Solve for \( \gamma(t) \in G, \mu(t) \in A, \nu(t) \in Y \)

\[
\begin{align*}
\nu_t &= F(v) - d[a(\mathbb{1})v]_{\mu}, \quad \nu(0) = u_0 \\
\gamma_t &= dL_{\gamma}(\mathbb{1})\mu, \quad \gamma(0) = 1 \\
0 &= \psi(v, \mu)
\end{align*}
\]  

(DAEV)

- related approach: Rowley, Kevrekidis, Marsden and Lust 2003
- The term \( d[a(\mathbb{1})v]_{\mu} \) in (DAEV) is obtained by applying \( \frac{d}{dg} \) to \( a(\gamma)a(g)v = a(L_{\gamma}g)v \) at \( g = \mathbb{1} \)

\[
a(\gamma)d[a(\mathbb{1})v]_{\mu} = d[a(\gamma)v]dL_{\gamma}(\mathbb{1})\mu \quad \text{for all } \mu \in A.
\]
Reaction diffusion systems in $\mathbb{R}^2$

\[
  u_t = \Delta u + f(u), \quad t \geq 0
\]
\[
  u(x, 0) = u_0(x), \quad x \in \mathbb{R}^2
\]

Action of Euclidean group

\[
  G = SE(2) = S^1 \times \mathbb{R}^2 \ni \gamma = (\phi, \tau)
\]
\[
  [a(\gamma)v](x) = v(R_{-\phi}(x - \tau))
\]

with the rotations

\[
  R_\phi = \begin{pmatrix}
    \cos(\phi) & -\sin(\phi) \\
    \sin(\phi) & \cos(\phi)
  \end{pmatrix}
\]

group operation

\[
  (\phi_1, \tau_1) \circ (\phi_2, \tau_2) = (\phi_1 + \phi_2, \tau_1 + R_{\phi_1} \tau_2)
\]
Reaction diffusion systems in $\mathbb{R}^2$

\[ u_t = \Delta u + f(u), \quad t \geq 0 \]
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symmetry terms (angular and translational velocities)

\[ d[a(\mathbb{1})v]\mu = \mu_1(yv_x - xv_y) + \mu_2v_x + \mu_3v_y \]
Quintic Ginzburg Landau equation, 2d

\[ u_t = \alpha \Delta u + \delta u + \beta |u|^2 u + \gamma |u|^4 u, \quad (x, y) \in \mathbb{R}^2, \quad u(x, y, t) \in \mathbb{C} \]

\[ \alpha = 0.5(1 + i), \quad \delta = -0.5, \quad \beta = 2.5 + i, \quad \gamma = -1 - 0.1i, \]
\[ a(\gamma)v(\xi) = e^{i\theta}v(R_{-\phi}(\xi - \tau)) \text{ for } \gamma = (\phi, \tau, \theta) \in G = SE(2) \times S^1, \]
Quintic Ginzburg Landau equation, 2d

\[ u_t = \alpha \Delta u + \delta u + \beta |u|^2 u + \gamma |u|^4 u \]
\[ + \mu_1 (yu_x - xu_y) + \mu_2 u_x + \mu_3 u_y + \mu_4 iu \]

\[ 0 = \langle yu_0, x - xu_0, y, u - u_0 \rangle_{L^2}, \quad 0 = \langle iu_0, u - u_0 \rangle_{L^2} \]
\[ 0 = \langle u_0, x, u - u_0 \rangle_{L^2}, \quad 0 = \langle u_0, y, u - u_0 \rangle_{L^2} \]

\[ \alpha = 0.5(1 + i), \quad \delta = -0.5, \quad \beta = 2.5 + i, \quad \gamma = -1 - 0.1i, \]
\[ a(\gamma)v(\xi) = e^{i\theta} v(R_{-\phi}(\xi - \tau)) \] for \( \gamma = (\phi, \tau, \theta) \in G = SE(2) \times S^1, \)

Computation with FEM package COMSOL Multiphysics, Neumann b.c.
Scroll waves in $\mathbb{R}^3$: CGL-system

$$u_t = \Delta u + (1 - |u|^2 - i|u|^2)u, \quad x \in \mathbb{R}^3, \; u(x, t) \in \mathbb{C}$$

Action of Euclidean group

$$G = SE(3) = SO(3) \ltimes \mathbb{R}^3, \quad \gamma = (R, \tau)$$

$$[a(\gamma)v](x) = v(R^{-1}(x - \tau))$$

group operation

$$\gamma \circ \tilde{\gamma} = (R\tilde{R}, \tau + R\tilde{\tau})$$
Scroll waves in $\mathbb{R}^3$: CGL-system

\[ u_t = \Delta u + (1 - |u|^2 - i|u|^2)u, \quad x \in \mathbb{R}^3, \ u(x, t) \in \mathbb{C} \]

Action of Euclidean group

\[ G = SE(3) = SO(3) \ltimes \mathbb{R}^3, \ \gamma = (R, \tau) \]

\[ [a(\gamma)v](x) = v(R^{-1}(x - \tau)) \]

group operation

\[ \gamma \circ \tilde{\gamma} = (R\tilde{R}, \tau + R\tilde{\tau}) \]

\[ \nu_t = \Delta \nu + (1 - |\nu|^2 - i|\nu|^2)\nu + \mu_4 \nu_{x_1} + \mu_5 \nu_{x_2} + \mu_6 \nu_{x_3} \]

\[ + \mu_1(\nu_{x_2}x_3 - \nu_{x_3}x_2) + \mu_2(\nu_{x_3}x_1 - \nu_{x_1}x_3) + \mu_3(\nu_{x_1}x_2 - \nu_{x_2}x_1) \]

corresponding phase conditions

Numerical solution with adaptation of ezscroll (Barkley ’97)

\( L_{x_i} = 40, \ \Delta x_i = 1, \ \Delta t = \frac{3}{8}10^{-3}, \) 19-point Laplacian,
boundary conditions: \( x, y \) - Neumann, \( z \) - periodic

initial function \( u_0(r, \varphi, z) = \exp\left(\frac{iz}{2\pi}\right)\frac{r}{40}(\cos(\varphi) + i\sin(\varphi)) \)
Scroll wave in 3d

initial cond., $x,y,x$-slices through origin

isosurface, $\text{Re}(u)=0$

solution at $t = 300$, $x,y,x$-slices through origin

isosurface, $\text{Re}(u)=0$
Relative equilibria and relative periodic orbits

**Definition:** A solution \( u(t) = a(\gamma(t))\bar{v}, \bar{v} \in Y, \gamma \in C^1(\mathbb{R}_+, G) \) of \( u_t = F(u) \) is called a relative equilibrium. Likewise, a solution \( u(t) = a(\gamma(t))\bar{v}(t) \) is called a relative periodic orbit if \( \bar{v}(t) \) has nontrivial period \( T > 0 \).

**Characterization:** Assume \( d[a(\mathbb{1})\bar{v}] : \mathcal{A} \mapsto X \) is one to one. Then \( a(\gamma(t))\bar{v} \) is a relative equilibrium iff there exists \( \bar{\mu} \in \mathcal{A} \) such that \( \bar{v} \) is a steady state of

\[
\begin{align*}

v_t &= F(v) - d[a(\mathbb{1})v]\bar{\mu} \\
(\text{PDAE})
\end{align*}
\]

and

\[
\begin{align*}

\gamma_t &= dL_{\gamma}(\mathbb{1})\bar{\mu} \\
(\text{RE})
\end{align*}
\]

Solution of (RE): \( \gamma(t) = \exp(t\bar{\mu})\gamma(0) \).
Asymptotic stability and freezing

Goal: Stability of relative equilibrium with asymptotic phase turns into Liapunov stability of the pair $(\bar{v}, \bar{\mu})$.

Stability with asymptotic phase:
A relative equilibrium $a(\gamma(t))\bar{v}$ is called stable with asymptotic phase if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for all solutions of $u_t = F(u)$ with $\|u(0) - \bar{v}\| \leq \delta$ there exists $\gamma_0(t) \in G$ satisfying

$$\|u(t) - a(\gamma_0(t))\bar{v}\| \leq \varepsilon, \quad \forall t \geq 0$$

$$\rightarrow 0 \quad \text{for } t \rightarrow \infty.$$

(Meta)Theorem: Stability with asymptotic phase holds if and only if $(\bar{v}, \bar{\mu}) \in Y \times A$ is an asymptotically stable equilibrium of

$$\begin{align*}
\nu_t &= F(\nu) - d[a(1)\nu]\mu, \quad \nu(0) = u_0 \\
\gamma_t &= dL_\gamma(1)\mu, \quad \gamma(0) = 1 \\
0 &= \psi(\nu, \mu)
\end{align*}$$

(DAEV)

in the Lyapunov sense for all consistent initial data.

Results: Traveling waves in 1D, rotating waves in 2D.
Stability of two-dimensional rotating patterns

with J. Lorenz DPDE, 2009.

For the reaction diffusion system in $\mathbb{R}^2$

$$u_t = A\Delta u + f(u), \quad u(x, 0) = u_0(x) \quad x \in \mathbb{R}^2, \ t \geq 0$$

(RD)

consider a rigidly rotating localized pattern

$$u(x, t) = \bar{v}(R_{-ct}x), \ c > 0, \ R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

(LP)

- $\sup_{|x| \geq R} |\bar{v}(x) - u_\infty| \to 0$ as $R \to \infty$,
- $f(u_\infty) = 0$ and $f'(u_\infty) \leq -\beta I < 0$,
- the eigenvalues $\pm ic$ with eigenvector $D_1 \bar{v} \pm iD_2 \bar{v}$ and 0 with eigenvector $D_\phi \bar{v}$ are simple for the linearized operator $\mathcal{L} = A\Delta + cD_\phi + f'(\bar{v})$ in $H^2_{\text{Eucl}} = \{ u \in H^2 : D_\phi u \in L^2 \}$,
- $\mathcal{L}$ has no further eigenvalues $s \in \mathbb{C}$ with $\text{Re} (s) \geq -\beta$. 

Nonlinear stability for rotating localized 2D-patterns

Theorem (with J. Lorenz)

Under the assumptions above there exists an $\varepsilon > 0$ such that for any solution of (RD) satisfying $\|u_0 - \bar{v}\|_{\mathcal{H}^2} \leq \varepsilon$ there is a $C^1$-function $\gamma(t) = (\theta(t), \tau(t)) \in SE(2)$ and $(\theta_\infty, \tau_\infty) \in SE(2)$ with

\[
\begin{align*}
\|u(\cdot, t) - a(\gamma(t))\bar{v}\|_{\mathcal{H}^2} &\leq C \exp\left(-\frac{\beta}{2}t\right)\|u_0 - \bar{v}\|_{\mathcal{H}^2} \\
|\theta(t) + ct - \theta_\infty| + |\tau(t) - \tau_\infty| &\leq C \exp\left(-\frac{\beta}{2}t\right)\|u_0 - \bar{v}\|_{\mathcal{H}^2}
\end{align*}
\]

From the proof:

- $\mathcal{L} = A\Delta + cD_\phi + f'(\bar{v})$ not sectorial, $\sigma_{\text{ess}}(\mathcal{L})$ contains curves, $s = \lambda_j(\kappa) + \text{inc}$, $n \in \mathbb{Z}$, $\kappa \in \mathbb{R}$, $\lambda_j(\kappa)$ ev. of $-\kappa^2 A + f'(u_\infty)$.
- Compact perturbation theorem for $C^0$-semigroups applies in $\mathcal{H}^2$ after splitting off the trivial eigenvalues $\pm ic, 0$.
- Use Sobolev and Gagliardo Nirenberg estimates in $\mathcal{H}^2$.

Remark: Convergence of freezing method still to be proved.
Essential spectrum

Linearization in polar coordinates

\[
\mathcal{L} = A \left( D_r^2 + \frac{1}{r} D_r + \frac{1}{r^2} D_\phi^2 \right) + cD_\phi + f'(\bar{v}(r, \phi))
\]

In the far field \((r = \infty)\): \(\mathcal{L}_{\text{far}} = AD_r^2 + cD_\phi + f'(u_\infty)\).

Find solutions of \(u_t = \mathcal{L}_{\text{far}} u\) that take the form

\[
u(r, \phi, t) = e^{st} e^{in\phi} e^{i\kappa r} v, \ r \geq 0, \ \phi \in [0, 2\pi] \text{ for some } v \in \mathbb{C}^m,
\]

\[
\det(-\kappa^2 A + \text{inc} + f'(u_\infty) - s) = 0 \quad \text{dispersion relation}
\]

**Theorem**

*If \(s\) satisfies the dispersion relation for some \(\kappa \in \mathbb{R}, \ n \in \mathbb{Z}\), then \(s \in \sigma_{\text{ess}}(\mathcal{L})\).*

**Method of proof:** For \(u_R(r, \phi) = \chi_R e^{i(n\phi + \kappa r)} v\)

\((v \text{ an eigenvector, } \chi_R \text{ a cut-off function})\) show

\[
\|(\mathcal{L} - s)u_R\|_{L^2} \leq C, \quad \|u_R\|_{L^2} \geq C\sqrt{R}
\]
Example: Quintic Ginzburg Landau

$$u_t = \alpha \Delta u + \delta u + \beta |u|^2 u + \gamma |u|^4 u, \quad (x, y) \in \mathbb{R}^2, \ t \geq 0.$$  

Infinitely many copies of two half lines

$$s = -\kappa^2 \alpha + \text{inc} + \delta, \quad s = -\kappa^2 \bar{\alpha} + \text{inc} + \bar{\delta}, \quad \kappa \in \mathbb{R}, \ n \in \mathbb{Z}.$$  

Data: $\alpha = \frac{1+i}{2}, \ \delta = -\frac{1}{2} < 0$

essential spectrum: $s = \text{inc} + \delta - \kappa^2 (\alpha_1 \pm i\alpha_2), \ \kappa \in \mathbb{R}, \ n \in \mathbb{Z}$
Generic picture of full spectrum

Essential spectrum, critical eigenvalues, and further isolated eigenvalues

Semigroup $e^{tL}$ is continuous but not analytic!
Part of numerical spectrum: 400 ev, system size $\approx 10^5$

$R=30$, $h_{\text{max}}=0.25$, $\text{neig}=400$

8 additional pairs of isolated eigenvalues.
Real parts of eigenfunctions

2 critical and 8 extra isolated eigenvalues, 2 'non-eigenvalues'
Freezing Multifronts and Multipulses


\[
\begin{align*}
    u_t & = u_{xx} + f(u), \quad x \in \mathbb{R}, t \geq 0, u(x, t) \in \mathbb{R}^m \\
    u(x, 0) & = u_0(x), \quad x \in \mathbb{R}
\end{align*}
\]

Ansatz for decomposition into single fronts

\[
    u(x, t) = \sum_{j=1}^{N} v_j(x - g_j(t), t), \quad g_j(t) \text{ position of } j\text{-front at time } t
\]

Take a bump function \( \varphi \in C^\infty(\mathbb{R}, \mathbb{R}) \) such that
\[
0 < \varphi(x) \leq 1 \quad \forall x \in \mathbb{R}
\]
and use a partition of unity depending on positions \( g_j \)

\[
Q_j(g, x) = \frac{\varphi(x - g_j)}{\sum_{k=1}^{N} \varphi(x - g_k)}, \quad x \in \mathbb{R}, j = 1, \ldots, N.
\]

Insert ansatz into the PDE.
Abbreviate \( v_k(\cdot) = v_k(\cdot - g_k(t), t) \) and find

\[
\begin{align*}
\dot{u}_t &= \sum_{j=1}^{N} \left[ v_{j,t}(\cdot) - v_{j,x}(\cdot)g_{j,t} \right] \\
&= \sum_{j=1}^{N} \left[ v_{j,xx}(\cdot) + Q_j(g, \cdot) f \left( \sum_{k=1}^{N} v_k(\cdot) \right) \right] \\
&= \sum_{j=1}^{N} \left[ v_{j,xx} + f(v_j) + Q_j(g, \cdot) \left( f \left( \sum_{k=1}^{N} v_k(\cdot) \right) - \sum_{k=1}^{N} f(v_k(\cdot)) \right) \right].
\end{align*}
\]

Require that the terms \[ \ldots \ldots \] in the first and in the third sum match.
Set

\[
\xi = x - g_j(t), \quad j = 1, \ldots, N
\]
\[
*_{kj} = \xi - g_k(t) + g_j(t) \quad j, k = 1, \ldots, N.
\]

Then a **sufficient** condition for the system to be satisfied is

\[
v_{j,t}(\xi, t) = v_{j,\xi}(\xi, t) + v_{j,\xi}(\xi, t)\mu_j(t) + f(v_j(\xi, t))
\]
\[
+ \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(*)_{kj}} \left[ f \left( \sum_{k=1}^N v_k(*_{kj}, t) \right) - \sum_{k=1}^N f(v_k(*_{kj}, t)) \right]
\]

\[
0 = \langle v_j(\cdot, t) - \hat{v}_j, \hat{v}_{j,x} \rangle_{L_2}, \quad v_j(\cdot, 0) = v_j^0(\cdot),
\]
\[
g_{j,t} = \mu_j, \quad g_j(0) = g_{j,0}.
\]

- **PDAE-System with nonlinear and nonlocal coupling.**
- **Decomposition is generally not unique!**
- **For fronts one needs a modified version.**
- **Solve the above system on a finite interval \([x_-, x_+]\).**
Nagumo equation

\[
u_t = u_{xx} + u(1 - u)(u - a), \quad x \in \mathbb{R}, t \geq 0, a = \frac{1}{4}\]

Recall the FitzHugh-Nagumo wave

\[
\begin{align*}
V_t &= \Delta V + V - \frac{1}{3}V^3 - R + \mu V_x, \\
R_t &= \phi(V + a - bR) + \mu R_x,
\end{align*}
\]

\( J = [0, 130], \Delta x = 0.5, \Delta t = 0.01, a = 0.7, b = 0.8, \phi = 0.08. \)

**Upwind/downwind for convective term \( \mu v_x \)**

\[ v_x \approx D_\pm v = \alpha D_+ v + (1 - \alpha) D_- v, \quad \alpha = (1 + e^{-\beta \mu})^{-1}, \quad \beta = 1 \]

**traveling vs. frozen**

\( \psi_{\text{orth}} \) vs. \( \psi_{\text{fix}} \) phase condition
Decompose and freeze

Theory (S. Selle 2009)
Asymptotic stability with asymptotic phase for the coupled system with nonlocal terms in case of weak interaction, i.e. initial data are close to a superposition of waves that are far apart and that are asymptotically stable individually.
Related work on the original equation by D. Wright 2008/09.
Summary

- Freezing allows to adaptively compute moving coordinate systems for equivariant PDEs
- Leads to (P)DAEs of index 1 or 2 with additional convective terms
- For parabolic systems in 1D the effects of the transformation $\text{PDE} \xrightarrow{\text{freezing}} \text{PAE} \xrightarrow{\text{discretization}} \text{DAE}$ have been analyzed
- Stability with asymptotic phase is converted into Lyapunov stability (proved for traveling waves in parabolic and certain hyperbolic systems, almost proved for 2D-rotating patterns in parabolic systems).
- Works for numerical examples in 1D-3D. Artificial convection can create problems for discretizations.
- Multifronts and multipulses with different speeds can be frozen independently (B., Selle, Thümler 2008), nonlinear stability theorem for the case of weak interaction (Selle 2009).
Perspectives

- Freezing of relative equilibria for different types of equations
  - Viscous conservation laws,
  - Nonlocal diffusion terms,
  - SPDEs (G. Lord, V. Thümmler)

- More general equivariance $b(\gamma)F(u) = F(a(\gamma)u)$ occurs when rescaling time, see Rowley et al. 2003

- Stability proofs for the PDAE formulation in dimensions $\geq 2$

- Systems with relative periodic orbits and their direct computation (Wulff, Schebesch 2006, Champneys, Sandstede 2007)

- Extension of the 'decompose and freeze' approach to multistructures in dimensions $\geq 2$. 
References


