Breakdown of analyticity: From rigorous results to numerics

Renato Calleja

Applied Mathematics Laboratory
Centre de Recherches Mathématiques

July 7th, 2010

Bifurcation Analysis and its Applications

Joint work with Rafael de la Llave

Outline

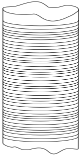
Smooth maps of $\mathbb{A} = \mathbb{R} \times \mathbb{T}$ given by $f(p, q) = (p', q')$

- Preserve area
- Twist $\partial q' / \partial p \geq \delta > 0$

- Appear in Celestial Mechanics, Plasma Physics.
- Admit a variational description.

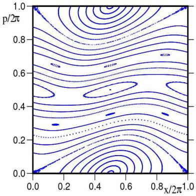
Integrable twist maps

$$p' = p$$
$$q' = q + \omega(p)$$



Standard map

$$p' = p + \frac{\lambda}{2\pi}\sin(2\pi q)$$
$$q' = q + p' \pmod 1$$



Poincaré map of
a conservative system.

Quasi-periodic solutions are orbits of the form

$$(q_n, p_n) = K(n\omega), \quad \omega \in \mathbb{R} \setminus \mathbb{Q}$$

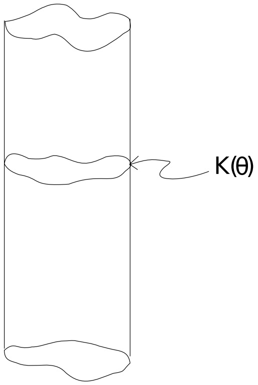
In such a case, we have

$$F \circ K(\theta) = K(\theta + \omega) \qquad (1)$$

We will assume

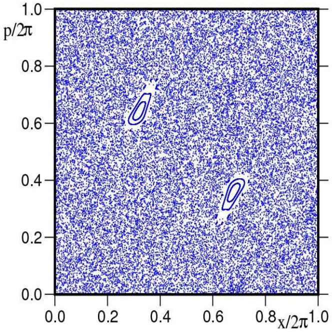
$$K(\theta + 1) = K(\theta) + (1, 0).$$

"non-contractible circles".



Conventional Wisdom

As we increase the perturbation, analytic Tori turn into Cantor sets until there is one "last circle".
When the last circle breaks down, there is "global chaos".

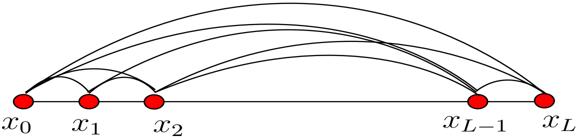


Formulation of the models

We consider models

$$\mathcal{S}(\{x_n\}) = \sum_{L\in\mathbb{N}} \sum_{k\in\mathbb{Z}} H_L(x_k, \dots, x_{k+L}) \qquad (2)$$

The meaning of $H_L(x_0, \dots, x_L)$

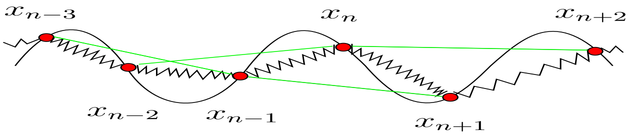


$$\frac{\partial \mathcal{S}}{\partial x_n} = 0 \Leftrightarrow \sum_{L} \sum_{\substack{k+j=n \\ j=0,\dots,L}} \partial_j H_L(x_k, \dots, x_{k+j}, \dots, x_{k+L}) = 0$$

Frenkel-Kontorova models

Consider $H_0(t) = \lambda V(t)$, $H_1(x, y) = \frac{1}{2}|x - y - a|^2$, and $H_L = \frac{A_k}{2}(x_0 - x_k - b_k)^2$ for $L \geq 2$.

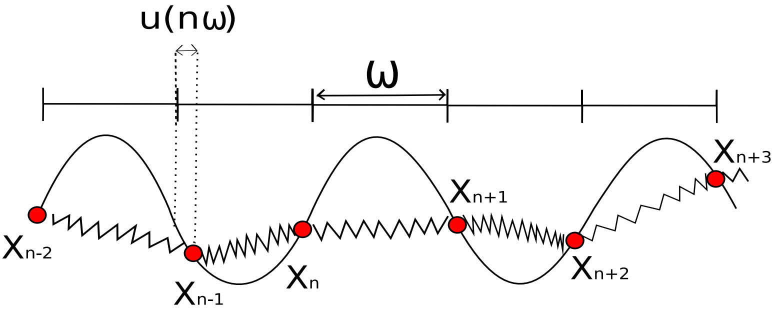
1. $A_k = 0$ for $k \geq 2$, Frenkel-Kontorova.
2. $A_2 = A$ and $A_k \equiv 0$ for $k \geq 3$.
3. $A_k = ak^\alpha$ (hierarchical model Dyson).



Equilibrium: $\sum_{k\in\mathbb{N}} A_k(x_{n-k} - 2x_n + x_{n+k}) - \lambda V'(x_n) = 0.$

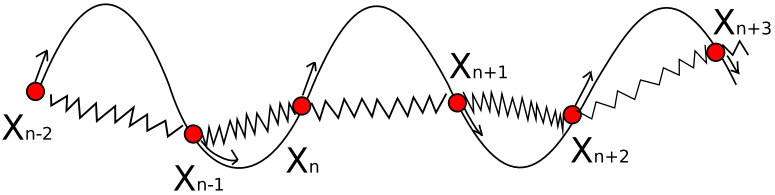
Hull functions

$$x_n = h(n\omega) = n\omega + u(n\omega), \quad n \in \mathbb{Z}, \omega \in \mathbb{R} \setminus \mathbb{Q}$$



Sliding

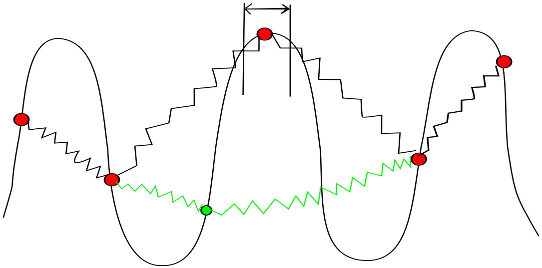
$$h(\theta) = \theta + u(\theta)$$



$$E[u](\theta) \equiv \sum_{L} \sum_{j=0}^{L} \partial_j H_L(\theta - j\omega + u(\theta - j\omega), \ldots, \theta + u(\theta), \ldots, \theta + (L-j)\omega + u(\theta + (L-j)\omega)) = 0 \quad (3)$$

Fix a Diophantine frequency

1. KAM theory ensures the existence of smooth quasi-periodic solutions for “quasi-integrable” system.
2. There are examples with no smooth equilibria.



The transition is very similar to the phase transition and there are scaling relations and Renormalization Group transformations.

- *Can we compute the boundary of existence of smooth solutions?*
- *What happens near the boundary?*

Our goal is to find algorithms to compute the boundary of analyticity

- efficient
- rigorously justified
- work in cases where there is no variational principle (e.g. Heisenberg Models, Non-Twist mappings)
- work in cases with no dynamical formulation(e.g. extended Frenkel-Kontorova Model).

A prototype *a posteriori* theorem

Theorem

$\mathcal{X}_0 \subset \mathcal{X}_1$ and $\mathcal{F} : \mathcal{U} \subset \mathcal{X}_0 \to \mathcal{X}_0$,
Computable functionals $f_1, ..., f_n : \mathcal{X}_0 \to \mathbb{R}^+$
Suppose that $\xi_0 \in \mathcal{X}_0$ with $\|\xi_0\|_{\mathcal{X}_0} \leq M_0$,

1. $\|\mathcal{F}(\xi_0)\|_{\mathcal{X}_0} < \varepsilon$
2. $f_1(\xi_0) \leq M_1, ..., f_n(\xi_0) \leq M_n$
3. $\varepsilon \leq \varepsilon^*(M_0, M_1, ..., M_n)$

Then there exists an $\xi^* \in \mathcal{X}_1$ such that $\mathcal{F}(\xi^*) = 0$ and

$$\|\xi_0 - \xi^*\|_{\mathcal{X}_1} \leq C\varepsilon$$

For numerics we use Sobolev Spaces.

Consequences of *a posteriori* theorems

- Local uniqueness and smoothness with respect to parameters.
- Validation of numerical algorithms producing a solution.
- Continuation methods of numerical analysis work.

Note we do not assume that the system is close to integrable

The proof of the existence theorem is based on a Nash-Moser Newton-like method.

The proofs that can be "followed" closely by numerics.

As is known since Kolmogorov, Newton can overcome the problem of "small divisors".

- $\mathcal{F}[\xi]$ is small,
- $\mathcal{F}[\tilde{\xi}] = \mathcal{F}[\xi] + D\mathcal{F}[\xi](\tilde{\xi} - \xi) + O(\|\tilde{\xi} - \xi\|^2)$,
- $\tilde{\xi} = \xi - D\mathcal{F}[\xi]^{-1}\mathcal{F}[\xi]$,
- $\|\mathcal{F}[\tilde{\xi}]\| \approx \|\mathcal{F}[\xi]\|^2$.

Nash–Moser Step

Nash–Moser:
$D\mathcal{F}[\xi]^{-1}$ is unbounded.

(bounded from one space to another, of less regular functions).

$$\tilde{\xi} = \xi - SD\mathcal{F}[\xi]^{-1}\mathcal{F}[\xi]$$

S is a “smoothing” operator with restores regularity.

N-M

- By careful adjusting the smoothing operators one can get the procedure to converge.
- One technical point is that one does not need a true inverse of $D\mathcal{F}[K]$ but an "approximate inverse" suffices.

Numerics

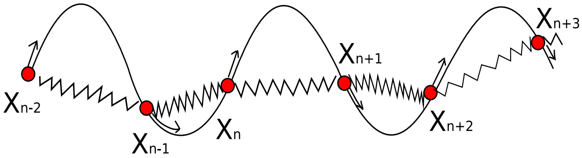
- Close to the Breakdown, "smoothing" helps avoiding spurious solutions.
- The approximate inverse helps produce fast algorithms.

Example: The Lagrangian version of the Standard map

$$E[u] \equiv u(\theta+\omega) + u(\theta-\omega) - 2u(\theta) + V'(\theta + u(\theta))$$

We make the correction $\tilde{u} = u + \Delta$, then Δ satifies the Newton Step equation

$$\Delta(\theta+\omega) + \Delta(\theta-\omega) - 2\Delta(\theta) + V''(\theta + u(\theta))\Delta(\theta) = -E[u]$$



Even in the simpler case $V = 0$
Difference equation with constant coefficients.

$$\Delta(\theta + \omega) + \Delta(\theta - \omega) - 2\Delta(\theta) = -E[u]$$

in Fourier space

$$2(\cos(2\pi\omega k) - 1)\hat{\Delta}_k = \hat{E}_k$$

Even in the simpler case $V = 0$

$$\Delta(\theta+\omega) + \Delta(\theta-\omega) - 2\Delta(\theta) = -E[u]$$

in Fourier space

$$\hat{\Delta}_k = \frac{\hat{E}_k}{2(\cos(2\pi\omega k) - 1)}$$

Small divisors!

ω should be diophantine.

Numerics:
Diagonal in Fourier Space, $O(N \log N)$ operations (from FFT).

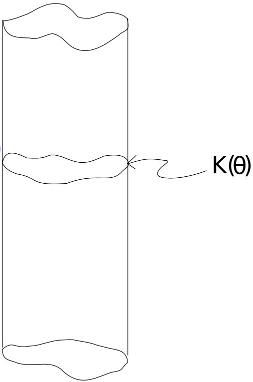
Example: Twist maps

F is a Twist map.

$$E[K](\theta) \equiv F(K(\theta)) - K(\theta + \omega)$$

$$\tilde{K} = K + \Delta$$

$$DF(K(\theta)\Delta(\theta) - \Delta(\theta + \omega) = -E[K](\theta)$$



In the
integrable case we have that

$$DF \circ K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We have a difference equation with constant coefficients.

$$\Delta_1(\theta) - \Delta_1(\theta+\omega) = -E_1[K](\theta) - \Delta_2(\theta)$$
$$\Delta_2(\theta) - \Delta_2(\theta+\omega) = -E_2[K](\theta)$$

$$\hat{\Delta}_{2,k} = \frac{-\hat{E}_{2,k}}{1 - e^{2\pi i k\omega}}$$

$$\hat{\Delta}_{1,k} = \frac{-\hat{E}_{1,k} - \hat{\Delta}_{2,k}}{1 - e^{2\pi i k\omega}}$$

Numerics:
Diagonal in Fourier Space, $O(N \log N)$ operations (from FFT).

Quasi-Newton Method

The Quasi-Newton Step consists in using geometric identities to find an approximate solution of the linearized equation using only

- Multiplications of functions,
- Differentiations of functions,
- Solving difference equations with constant coefficients.

The same geometric cancellations above can be used to obtain **fast** and stable numerical methods.

$$O(N \log N)$$

Algorithm for twist maps

1. $E[K](\theta) = F(K(\theta)) - K(\theta + \omega)$
2. Newton's equations
   $DE[K]\Delta(\theta) = DF(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) = -E[K](\theta)$
3. $M(\theta) = (DK(\theta) \quad J^{-1}(K(\theta))DK(\theta)N(\theta))$
4. $\Delta(\theta) = M(\theta)W(\theta)$
5. Solve for W from
   $$\begin{pmatrix} I_n & S(\theta) \\ 0 & I_n \end{pmatrix} W(\theta) - W(\theta + \omega) = -M^{-1}(\theta + \omega)E[K](\theta).$$
   (FFT + (*, /) + FFT) gives $O(N \log N)$.
6. Improved solution $\tilde{K}(\theta) = K(\theta) + M(\theta)W(\theta)$.

Algorithm for Statistical Mechanics Models

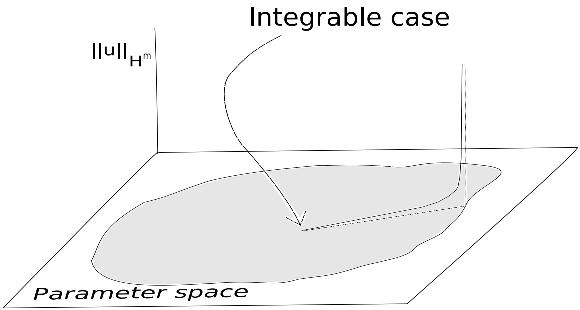
1. $$E[u](\theta) \equiv \sum_{L} \sum_{j=0}^{L} \partial_j H_L(\theta - j\omega + u(\theta - j\omega), \ldots,$$
$$\theta + u(\theta), \ldots, \theta + (L-j)\omega + u(\theta + (L-j)\omega)) = 0$$
2. Newton's equations $DE[u]v(\theta) = -E[u](\theta)$
3. Transformation $v(\theta) = h'(\theta)w(\theta)$
4. Modified Newton's equations
$h'(\theta)(DE[u]v)(\theta) - v(\theta)(DE[u]h')(\theta) = -h'(\theta)E[u](\theta)$
5. $\mathcal{L}_1[(\mathcal{C}_{0,1,1} + \mathcal{G})\mathcal{L}_{-1}w] = -h'E[u]$
(FFT + CG + FFT) gives $O(N \log N)$.
6. Improved solution $\tilde{u}(\theta) = u(\theta) + h'(\theta)w(\theta)$

There are versions for:

- Dissipative discrete systems (Quasi-periodic attractors).
- Flows (KAM tori).
- Dissipative flows(Quasi-periodic attractors).

In practice, the functionals we need to check are:

- Non-degeneracy of the problem
- That the approximate solution is rather regular
- In principle, could be transformed into computer assisted proofs.



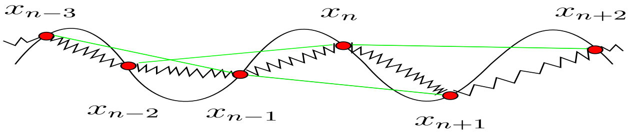
Choose a path in the parameter space starting in the integrable case.
Initialize the parameters and the solution at integrable
Repeat
    Increase the parameters along the path
    Run the iterative step
    If (Iterations do not converge)
        Decrease the increment in parameters
    Else (Iteration success)
        Record the values of the parameters
        and the Sobolev norm of the solution.
Until Sobolev norm too large

We have **existence and uniqueness**.
Then, we know the breakdown occurs.

Frenkel-Kontorova with first and second neighbor interaction

If we consider first and second neighbor interactions the corresponding Euler-Lagrange equations are

$$\begin{aligned} E[u](\theta) \equiv\ & u(\theta+\omega)+u(\theta-\omega)-2u(\theta) \\ +\ & A(u(\theta+2\omega)+u(\theta-2\omega)-2u(\theta))+V'(\theta+u(\theta)) \end{aligned} \tag{4}$$



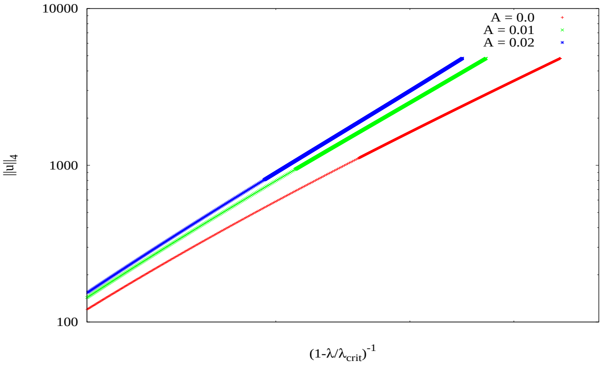


Figure: H^4 norm for fixed A

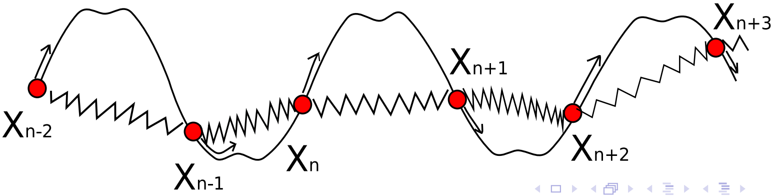
Multi-harmonic Standard map

If we consider first and second neighbor interactions the corresponding Euler-Lagrange equations are

$$E[u](\theta) = u(\theta+\omega) + u(\theta-\omega) - 2u(\theta) + V'(\theta + u(\theta)) \quad (5)$$

$$V'(x) = \frac{\varepsilon_1}{2\pi}\sin(2\pi x) + \frac{\varepsilon_2}{4\pi}\sin(4\pi x)$$

Fact:
Plethora of periodic orbits. (Lomelí and C. 2006)



Smooth regions and complicated regions

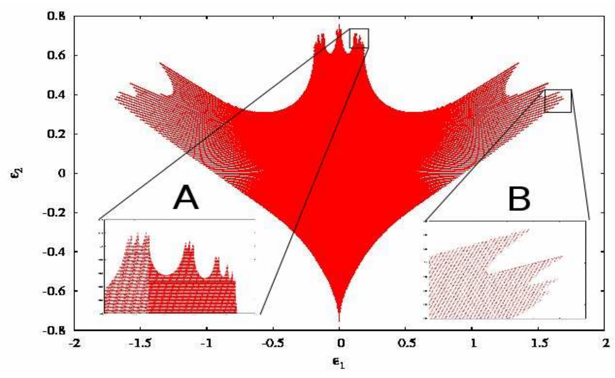


Figure: Neighborhood of analyticity in the parameter space $\varepsilon_1, \varepsilon_2$.

Empirical findings

- In several parts of the boundary one can fit

$$\|u_\lambda\|_{H^m} \approx \alpha(\lambda - \lambda_{crit})^{-0.987(m-\beta)}$$

- The boundary has smooth components and folds.
- As $\lambda \to \lambda_{crit}$ the conjugacy u_λ develops a self similar structure.
- The scaling law remains true even for models with long range interactions.
- It breaks down for infinite range.

Example: the dissipative standard map

(Joint work with A. Celletti)

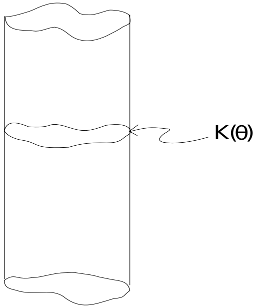
F_μ is given by

$$y_{n+1} = \lambda y_n + \mu + \varepsilon V'(x_n)$$
$$x_{n+1} = x_n + y_{n+1}$$

$$E[K] \equiv F_\mu \circ K - K \circ T_\omega$$

$$\tilde{K} = K + \Delta$$

$$DF_\mu \circ K \Delta - \Delta \circ T_\omega = -E[K]$$



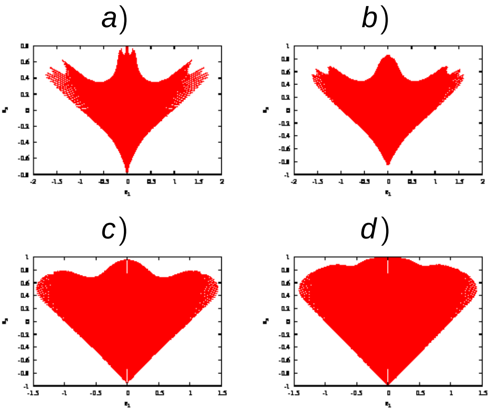


Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi}\sin(2\pi x) + \frac{\varepsilon_2}{4\pi}\sin(4\pi x)$. $a)$ $\lambda = 0.9$, $b)$ $\lambda = 0.5$, $c)$ $\lambda = 0.1$, $d)$ $\lambda = 0.01$.

Thank you.