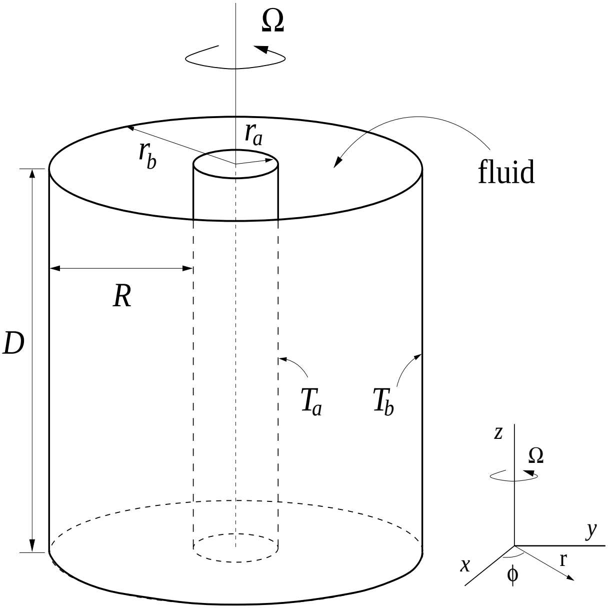
Flow transitions in a differentially heated rotating channel of fluid

Greg Lewis
with
Matt Hennessy

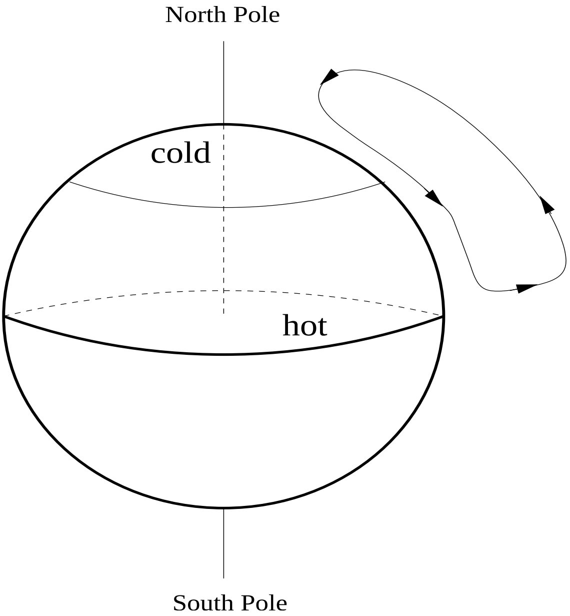
University of Ontario Institute of Technology

BAA, July 8, 2010

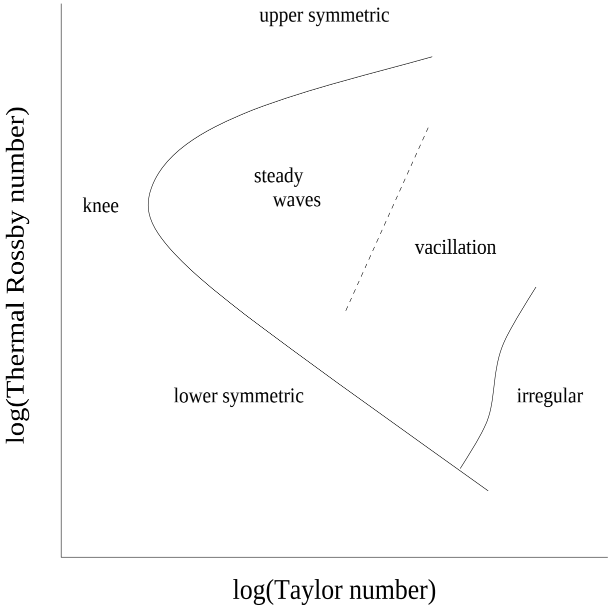
A differentially heated rotating annulus



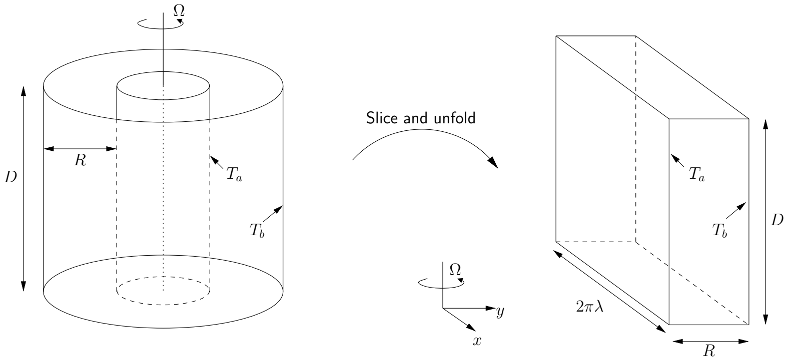
A differentially heated rotating planet



Regime diagram



Simplification: Fluid Channel



Symmetry

- new reflection symmetry:

$$[\mathbf{R}U](x,y,z) =$$

$$[-\mathbf{u}(2\pi\lambda - x, a+b-y, D-z), -T(2\pi\lambda - x, a+b-y, D-z)]$$

- with translational symmetry

$$[\mathbf{T}_l U](x,y,z) = [\mathbf{u}(x+l,y,z), T(x+l,y,z)].$$

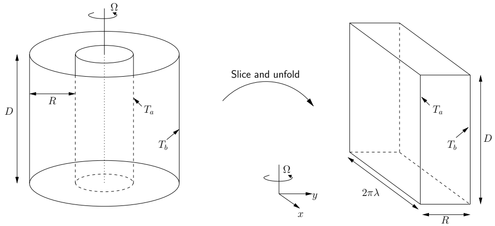
- now have $O(2)$ symmetry
- primary transition is a steady-state bifurcation to stationary waves
- except that transition to vacillation will correspond to a Hopf bifurcation

Overview of primary transition in channel

- use linear stability to locate transition
- look at steady-state mode-interactions
- centre manifold reduction and normal forms are used to deduce the form of the bifurcation
- discuss numerics

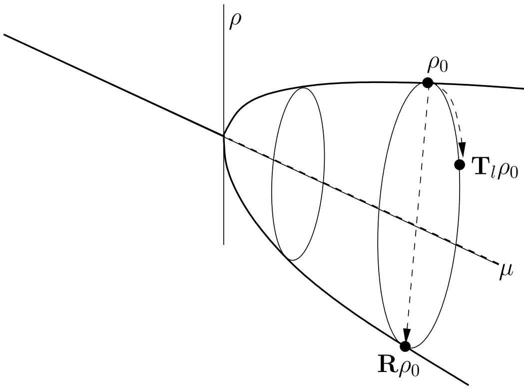
Model of Channel

- Navier-Stokes equations in three spatial dimensions
- In the Boussinesq approximation
- Assume channel is periodic in x
- Rotating frame of reference
- No-slip boundary conditions
- Insulating rigid top and bottom
- Parameters chosen to mimic previous annulus studies



Steady-state (pitchfork) bifurcations in $O(2)$

- 2 real eigenvalues vanish as a parameter is varied
- 1 zero eigenvalue corresponds to a wave with wave number m, and the other to $-m$
- bifurcation to a group orbit (of stationary waves)



Codimension-2 bifurcations (mode interactions)

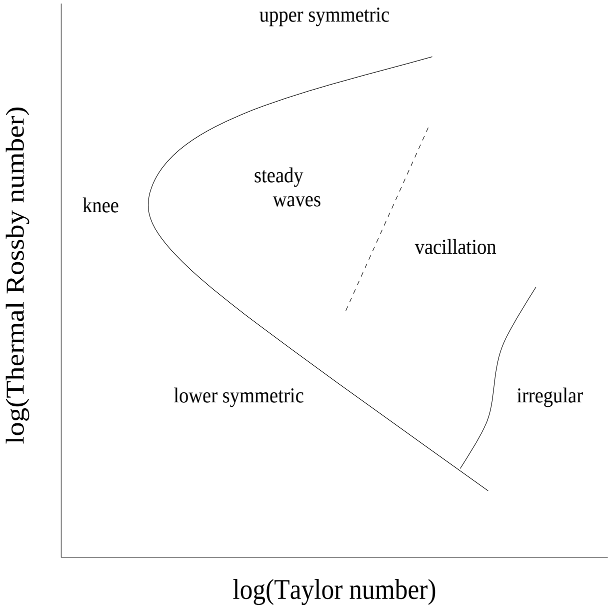
- 2 pairs of real eigenvalues vanish
- each pair corresponds to a different wave number m
- interesting dynamics
- need two parameters: Ω and ΔT

Summary of analysis

1. Trace out transition (bifurcation) curve
2. Locate the mode-interaction points (along transition curve)
3. Use center manifold reduction and write coefficients of normal form equations in terms of :
   - basic state
   - eigenfunctions
   - Taylor coefficients of the center manifold function
4. Numerically approximate these unknown functions

- Combination of numerical and analytical methods leads to approximations of 'normal form coefficients'

Regime diagram



Numerical approximations

- Approximate systems of 3 steady PDEs in 2 spatial dimensions
- Discretize on an $N \times N$ grid
  - 2nd-order centered finite differences
- Boundary layers in steady flow
  - transform non-uniform grid to uniform grid
- Use PETSc
  - sparse matrix storage and parallelization

Eigenvalue approximation

- Linearize about steady solution
- Assume eigenfunctions have the form

$$\Phi(x, y, z) = \hat{\Phi}_m(y, z)e^{imx/\lambda}$$

- get a series of generalized eigenvalue problems

$$\lambda \mathbf{B}_m \hat{\Phi}_m = \mathbf{A}_m \hat{\Phi}_m$$

- discretization leads to matrix eigenvalue problems

Computation of transition curve

- can find transition by solving the nonlinear equations for the basic state

$$\mathbf{f}(\mathbf{u}, T) = \mathbf{0}$$

together with an additional condition

$$s = 0$$

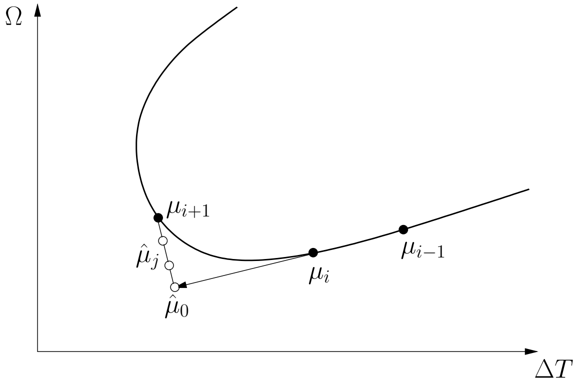
where s is given by

$$\begin{pmatrix} A_m & u \\ v^T & 0 \end{pmatrix} \begin{pmatrix} q \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

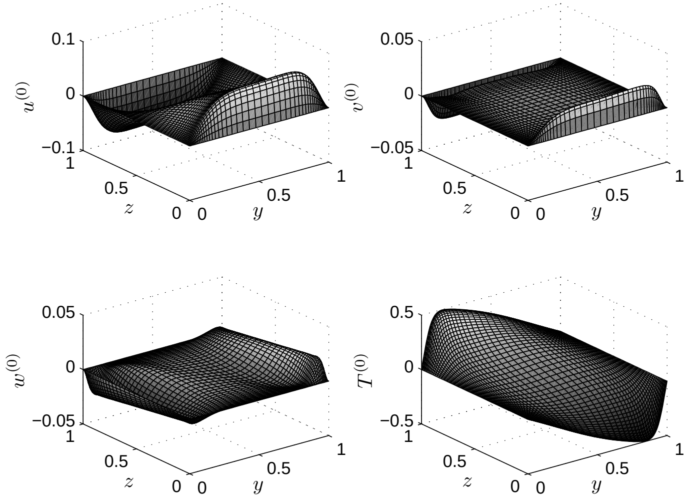
A_m results from the discretization of $\mathbf{A}_m$, where u and v can be some random vector (not in the range of the operator and the adjoint operator, respectively)

- for each m, use secant method to locate zero of s

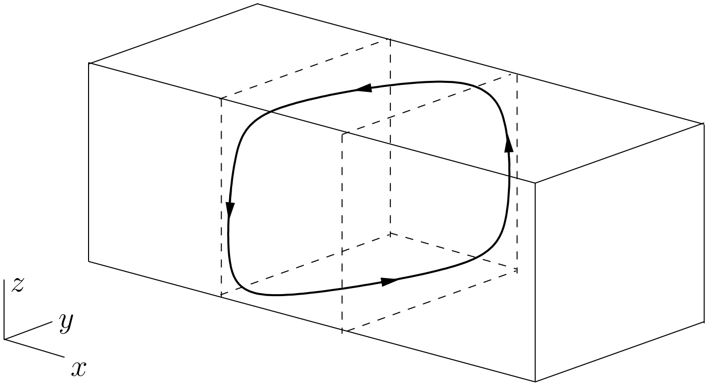
The Primary Transition



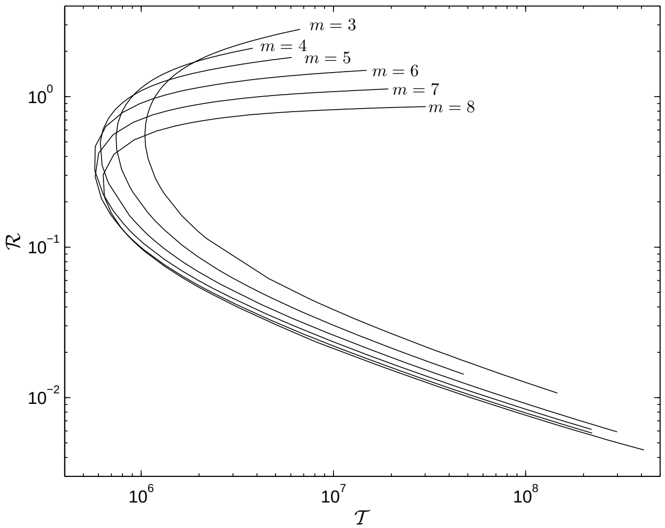
The Basic State (Steady, Uniform Flow)



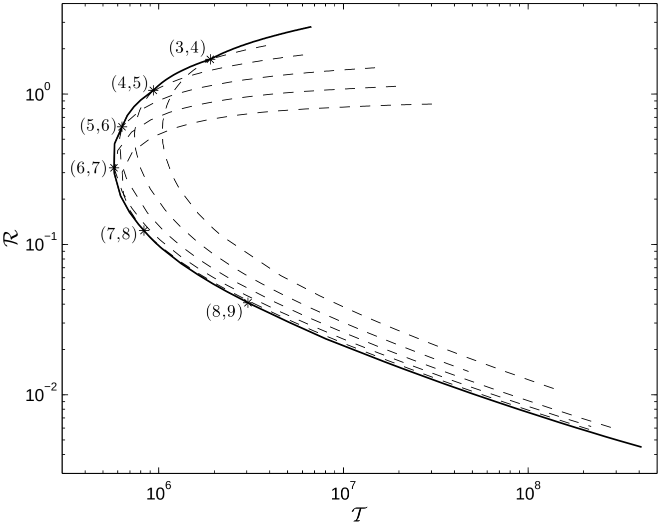
The Basic State (Steady, Uniform Flow)



The Primary Transition

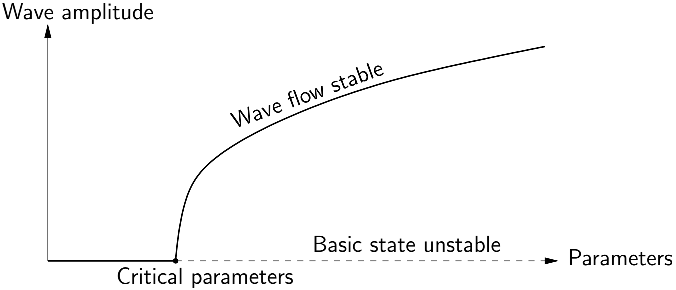


The Primary Transition



Primary transition dynamics

Transition curve composed of supercritical pitchfork bifurcations



Double pitchforks at isolated points along the transition curve

- Correspond to mode interactions

Steady-state mode-interaction: Notation / Definitions

- At Ω_0 and ΔT_0, the linearization about the steady solution has
  - two pairs of zero eigenvalues μ_j
  - with corresponding eigenfunctions: Φ_1, $\overline{\Phi}_1$, Φ_2, $\overline{\Phi}_2$
  - with form $\Phi_j(x, y, z) = \hat{\Phi}_j(y, z)e^{im_j x/\lambda}$
  - and all other eigenvalues have negative real part
- write the dependent variables as

$$U = z_1\Phi_1 + \overline{z}_1\overline{\Phi_1} + z_2\Phi_2 + \overline{z}_2\overline{\Phi_2} + \Psi$$

Reduced equations

Equation on the centre manifold

$$\begin{aligned}\dot{z}_1 &= \mu_1 z_1 + g_{11} z_1^2 \bar{z}_1 + g_{12} z_1 z_2 \bar{z}_2 + q_1 \bar{z}_1^{m_2-1} z_2^{m_1} + \ldots \\ \dot{z}_2 &= \mu_2 z_2 + g_{21} z_1 \bar{z}_1 z_2 + g_{22} z_2^2 \bar{z}_2 + q_2 z_1^{m_2} \bar{z}_2^{m_1-1} + \ldots\end{aligned}$$

where all coefficients are real.

- Write in polar coordinates ($z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$) and introduce a 'mixed phase' $\psi = m_2\theta_1 - m_1\theta_2$, after scaling get:

$$\begin{aligned}\dot{\rho}_1 &= \mu_1\rho_1 + a\rho_1^3 + b\rho_1\rho_2^2 + q_1'\rho_1^{m_2-1}\rho_2^{m_1}\cos\psi + \ldots \\ \dot{\rho}_2 &= \mu_2\rho_2 + c\rho_1^2\rho_2 + d\rho_2^3 + q_2'\rho_1^{m_2}\rho_2^{m_1-1}\cos\psi + \ldots \\ \dot{\psi} &= -\left(m_2 q_1'\rho_2^2 + m_1 q_2'\rho_1^2\right)\rho_1^{m_2-2}\rho_2^{m_1-2}\sin\psi + \ldots\end{aligned}$$

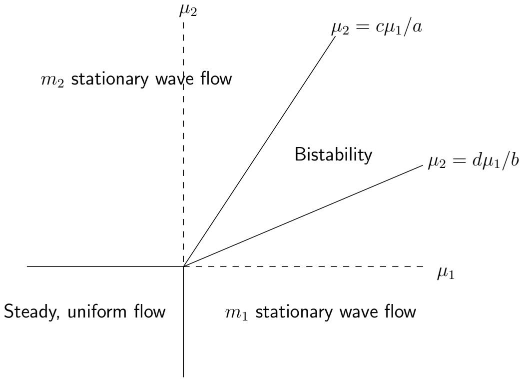
where and the eigenvalues are written as:

- $\mu_j = \mu_j(\Omega, \Delta T)$

- the coefficients a, b, c, d, q_1' and q_2' are functions of
  - the axisymmetric solution
  - the eigenfunctions
  - certain Taylor coefficients of the center manifold function, H

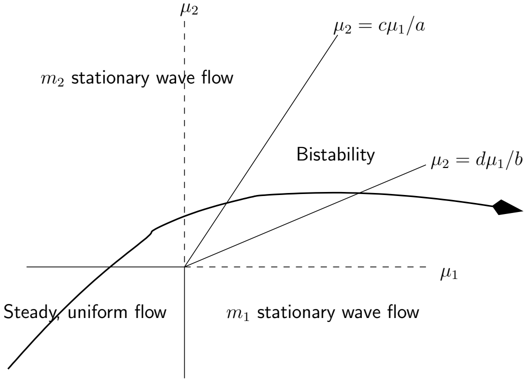
Mode interaction points

- Occur at intersection of neutral stability curves (m_1, m_2)
- Four distinct regions of fluid dynamics near each bifurcation:



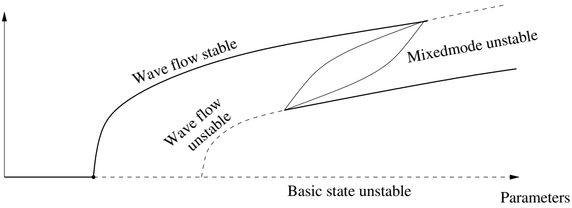
Mode interaction points

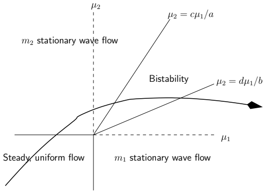
- Consider such a path through parameter space:



Mode interaction points

- Such a path through parameter space leads to the following bifurcation diagram:





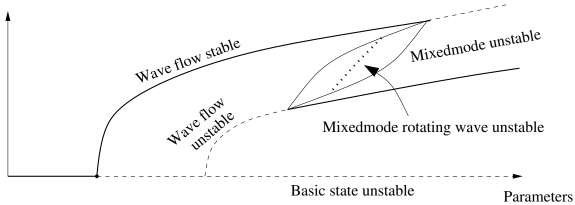
Reduced equations

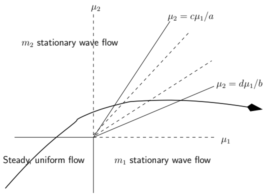
- Equations on the centre manifold (in polar form), where the 'mixed phase' $\psi = m_2\theta_1 - m_1\theta_2$

$$\begin{aligned}
\dot{\rho}_1 &= \mu_1\rho_1 + a\rho_1^3 + b\rho_1\rho_2^2 + q_1'\rho_1^{m_2-1}\rho_2^{m_1}\cos\psi + \dots \\
\dot{\rho}_2 &= \mu_2\rho_2 + c\rho_1^2\rho_2 + d\rho_2^3 + q_2'\rho_1^{m_2}\rho_2^{m_1-1}\cos\psi + \dots \\
\dot{\psi} &= -\left(m_2 q_1'\rho_2^2 + m_1 q_2'\rho_1^2\right)\rho_1^{m_2-2}\rho_2^{m_1-2}\sin\psi + \dots
\end{aligned}$$

Mode interaction points

- Depending on values of q'_1 and q'_2 may have:





Next steps

Investigate transition from stationary waves to vacillating waves

- corresponds to a Hopf bifurcation
- need to continue steady solutions of Navier-Stokes in 3 spatial dimensions
- need to locate bifurcation

Use:

- vorticity to eliminate pressure:

$$\begin{aligned} 0 &= \nu\nabla^2\omega + (\omega\cdot\nabla)\,\mathbf{u} - (\mathbf{u}\cdot\nabla)\,\omega \\ 0 &= \nabla^2\mathbf{u} + \nabla\times\omega \end{aligned}$$

- PETSc

Significantly more computationally challenging

Next steps

- Have successfully computed stationary wave solutions
  - Netwon's method
  - GMRES with ILU preconditioning for linear systems

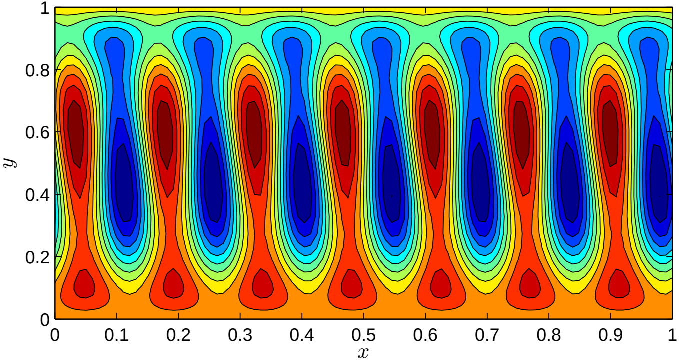


Figure: Temperature field of stationary wave solution at mid-depth

Computation of Hopf curve

- Look for parameter values such that

$$\mathbf{J} = \mathbf{L} + i\omega\mathbf{I}$$

is singular

- Solve the nonlinear equations for the basic state

$$\mathbf{f}(\mathbf{u}, T) = \mathbf{0}$$

together with an additional condition

$$s = 0$$

where s is given by

$$\begin{pmatrix} \mathbf{J} & u \\ v^T & 0 \end{pmatrix} \begin{pmatrix} q \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where u and v can be some random vectors (not in the range of the operator and the adjoint operator, respectively)

- use Broyden's method to locate zero of $s = s(p, \omega)$

Summary

- Computed primary flow transition
- Stationary waves equilibrate at transition (supercritical pitchfork)
- Mode interactions at intersection of neutral stability curves
- Channel shows a remarkable similarity to the annulus
- Suggests curvature of geometry not a significant factor
- Future/Current work:
  - Compute bifurcations from stationary waves : vacillation and beyond