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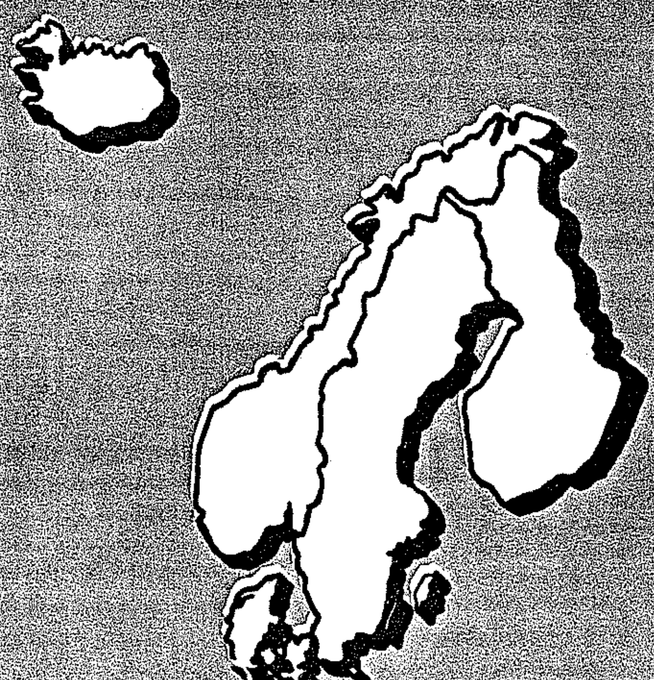

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SOME STABILITY THEOREMS FOR FINITE DIFFERENCE COLLOCATION METHODS ON NONUNIFORM MESHES

EUSEBIUS J. DOEDEL

Abstract.

A class of compact finite difference collocation methods is shown to be stable on general nonuniform meshes without the assumption of a bounded mesh ratio. Some error estimates are included.

1. Introduction.

We consider the convergence of certain discrete approximations to boundary value problems in ordinary differential equations. The approximations treated here have been studied previously in [2], [6] and [7] where, as in this paper, they are viewed as collocation procedures. Equivalently [2] the methods can be interpreted as finite difference approximations. This is the main approach in [1] and [5]. Convergence proofs of the methods can be based on the general theories of [4] and e.g. [3], [8], provided the mesh is uniform [1]. Stability (and hence convergence) on nonuniform meshes is difficult if not impossible to establish in general, except if the approximations are as compact as possible in the sense of [4]. Such stability results are contained in [5], subject to certain local and even global mesh ratio restrictions. The main purpose of this work is to show that these restrictions are not necessary. It is also shown that, under some added continuity assumptions, even the local placement of the collocation points can be quite arbitrary.

Collocation methods utilizing more conventional approximating spaces have been extensively studied elsewhere, see e.g. [9], but the analytical techniques used there do not immediately apply to the spaces in this work. This is due to the multivalued nature of the approximants and the fact that matching conditions take the place of more customary continuity requirements.

2. Stability results.

Consider the nth order equation

$$\text{(2.1a)} \qquad Ly \equiv y^{(n)}(x) + \sum_{l=0}^{n-1} a_l(x) y^{(l)}(x) = f(x), \qquad 0 \leqq x \leqq 1\,,$$

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subject to boundary conditions of the form

$$B_k y \equiv \sum_{l=0}^{n-1} \alpha_{k,l}(0) y^{(k)}(0) + \sum_{l=0}^{n-1} \alpha_{k,l}(1) y^{(k)}(1) = \beta_k, \quad 1 \leqq k \leqq n. \tag{2.1b}$$

Reference will also be made to the corresponding homogeneous problem

$$Ly = 0, \quad 0 \leqq x \leqq 1, \tag{2.2a}$$

subject to

$$B_k y = 0, \quad 1 \leqq k \leqq n. \tag{2.2b}$$

We approximate (2.1) by a class of generalized finite difference methods. These methods can be interpreted as collocation methods in the following manner. Introduce a mesh $\{0 = x_0 < x_1 < \ldots < x_J = 1\}$, with $h_j \equiv x_j - x_{j-1}$, $h \equiv (h_1, h_2, \ldots, h_J)$ and $|h| \equiv \max_j h_j$. To each meshpoint x_j, $(0 \leqq j \leqq J-n)$, associate a polynomial $p_j \in P_{n+m-1}$. (More accurately: $p_{h,j}(x)$.) We think of the interval $[x_j, x_{j+n}]$ as the domain of p_j. Let $\boldsymbol{p}_h \equiv \{p_j\}_{j=0}^{J-n}$. Thus $\boldsymbol{p}_h(x)$ is defined on $[0, 1]$. Note, however, that $\boldsymbol{p}_h$ is not single valued. The discrete method now consists of finding $\boldsymbol{p}_h$ satisfying the collocation equations

$$Lp_j(z_{j,i}) = f(z_{j,i}), \quad 1 \leqq i \leqq m, \ 0 \leqq j \leqq J-n, \tag{2.3}$$

where for each j the $z_{j,i}$ are distinct points in $[x_j, x_{j+n}]$. In addition $\boldsymbol{p}_h$ is required to satisfy the matching conditions

$$p_j(x_{j+i}) = p_{j+1}(x_{j+i}), \quad 1 \leqq i \leqq n, \ 0 \leqq j \leqq J-n-1, \tag{2.4}$$

as well as the boundary conditions

$$B_k \boldsymbol{p}_h = \beta_k, \quad 1 \leqq k \leqq n. \tag{2.5}$$

For each fixed mesh let $P_h^{n,m}$ denote the space of all $\boldsymbol{p}_h$ satisfying (2.4), and define

$$\|\boldsymbol{p}_h\|_k \equiv \max_{0 \leqq l \leqq k} \ \max_{0 \leqq j \leqq J-n} \ \max_{[x_j, x_{j+n}]} |p_j^{(l)}(x)|.$$

Also for $p \in C^k[0,1]$ and $\boldsymbol{p}_h \in P_h^{n,m}$ it will be convenient to use the notation

$$\|\boldsymbol{p}_h - p\|_k \equiv \max_{0 \leqq l \leqq k} \ \max_{0 \leqq j \leqq J-n} \ \max_{[x_j, x_{j+n}]} |p_j^{(l)}(x) - p^{(l)}(x)|.$$

Indeed, the above could serve as norm in an appropriate Banach space structure. However, such a formal set-up is not strictly necessary in subsequent analysis.

First we need the following:

LEMMA 2.1. *Let $\{h^\nu\}_{\nu=1}^{\infty}$ be a sequence of meshes with $|h^\nu| \to 0$ as $\nu \to \infty$. For each ν let $\boldsymbol{p}_{h^\nu} \in P_{h^\nu}^{n,m}$, $\|\boldsymbol{p}_{h^\nu}\|_n = 1$. Then there is a subsequence $\{\boldsymbol{p}_{h^{\nu_k}}\}_{k=1}^{\infty}$ and a function $p \in C^{n-1}[0,1]$, such that $\|\boldsymbol{p}_{h^{\nu_k}} - p\|_{n-1} \to 0$ as $k \to \infty$.*

REMARK. For notational simplicity we do not distinguish between sequence and subsequence.

PROOF. By simply restricting the domains of each of the component polynomials $\{p_j\}_{j=0}^{J-n-1}$ of p_{h^ν} to $[x_j, x_{j+1}]$, we can extract a single valued continuous function $\tilde{p}_{h^\nu}$ from each p_{h^ν}. Also $\|p_{h^\nu}\|_n = 1$ implies in particular that

$$\max_j \max_{[x_j, x_{j+n}]} |p_j'(x)| \leqq 1 .$$

Thus the sequence $\{p_{h^\nu}\}_{\nu=1}^{\infty}$ is equicontinuous and therefore has a subsequence $p_{h^\nu} \to p \in C[0,1]$. In fact, using the boundedness of the p_j', it is easy to see that the multivalued function p_{h^ν} converges to p, i.e.,

$$\|p_{h^\nu} - p\|_0 \to 0 \quad \text{as } \nu \to \infty .$$

We want to show that $p \in C^1[0,1]$. Define

$$q_{h^\nu} \equiv p_{h^\nu}' \equiv \{p_j'\}_{j=0}^{J-n} .$$

From Rolle's theorem and the matching conditions (2.4), it follows that every two consecutive component polynomials p_{j-1} and p_j of a given p_{h^ν} satisfy $p_{j-1}'(\xi_j) = p_j'(\xi_j)$ for some $\xi_j \in (x_j, x_{j+1})$. Therefore from each q_{h^ν} we can extract a single-valued continuous function $\tilde{q}_{h^\nu}$. The sequence $\{\tilde{q}_{h^\nu}\}_{\nu=1}^{\infty}$ is equicontinuous since $\max_j \max_{[x_j, x_{j+n}]} |q_j'(x)| \leqq 1$. Hence, (after relabelling everything) there is a subsequence $\tilde{q}_{h^\nu} \to q \in C[0,1]$. Again convergence of the multivalued q_{h^ν} to q follows from the boundedness of the q_j'. Thus $\|q_{h^\nu} - q\|_0 \to 0$ as $\nu \to \infty$.

We contend that $q = p'$. Let $x \in [0,1]$. Then

$$\int_0^x \tilde{q}_{h^\nu} = \int_0^{\xi_1} \tilde{q}_{h^\nu} + \sum_{j=1}^{j[x]-1} \int_{\xi_j}^{\xi_{j+1}} \tilde{q}_{h^\nu} + \int_{\xi_{j[x]}}^x \tilde{q}_{h^\nu} .$$

(Here $j[x]$ is the largest j for which $\xi_{j[x]} < x$). By definition of $\tilde{q}_{h^\nu}$ this implies

$$\begin{aligned}
\int_0^x \tilde{q}_{h^\nu} &= \int_0^{\xi_1} p_0' + \sum_{j=1}^{j[x]-1} \int_{\xi_j}^{\xi_{j+1}} p_j' + \int_{\xi_{j[x]}}^x p_{j[x]}' \\
&= p_{j[x]}(x) - \sum_{j=1}^{j[x]} [p_j(\xi_j) - p_{j-1}(\xi_j)] - p_0(0) \\
&= \tilde{p}_{h^\nu}(x) - \sum_{j=1}^{j[x]} (\xi_j - x_j)[q_j(\eta_{j,1}) - q_{j-1}(\eta_{j,2})] - \tilde{p}_{h^\nu}(0) ,
\end{aligned}$$

where $\eta_{j,1}, \eta_{j,2} \in (x_j, \xi_j)$.

It follows that

$$\begin{aligned}
\left| \int_0^x \tilde{q}_{h^\nu} - [p_{h^\nu}(x) - p_{h^\nu}(0)] \right| &\leqq \max_j |q_j(\eta_{j,1}) - q_{j-1}(\eta_{j,2})| \\
&= \max_j |(\eta_{j,1} - \xi_j) q_j'(\sigma_{j,1}) - (\eta_{j,2} - \xi_j) q_{j-1}'(\sigma_{j,2})| \leqq 2|h^\nu| ,
\end{aligned}$$

with $\sigma_{j,i} \in (\eta_{j,i}, \xi_j)$, $i=1,2$. Letting $\nu \to \infty$ and differentiating gives $p'(x)=q(x)$. It also follows that $\|p_{h^\nu}-p\|_1 \to 0$ as $\nu \to \infty$.

Using Rolle's theorem and the matching conditions (2.4) we can repeat the argument above another $n-2$ times, which finally yields the conclusion of the lemma. ∎

We can now prove:

THEOREM 2.2. *Let the homogeneous problem* (2.2a, b) *only admit the zero solution. Let* $a_l \in C[0,1]$, $l=0,1,\ldots,n-1$, *and assume that* $\min_{i_1 \neq i_2} |z_{j,i_1}-z_{j,i_2}| \geqq c|x_{j+n}-x_j|$ *for some constant c that is independent of j and h. Then there exist positive constants K and δ, such that the equations* (2.3) *and* (2.5) *admit a unique solution* $p_h \in P_h^{n,m}$ *and such that*

$$\|p_h\|_n \leqq K\left\{\max_{i,j} |f(z_{j,i})| + \max_l |\beta_l|\right\}, \tag{2.6}$$

whenever $|h| \in (0,\delta]$.

PROOF. (i) If (2.3), (2.5) do not have a unique solution $p_h \in P_h^{n,m}$ for all small $|h|$, then, since $\dim P_h^{n,m} = m(J-n+1)+n$ equals the number of equations in (2.3), (2.5), we can find a sequence of meshes $\{h^\nu\}_{\nu=1}^\infty$, with $|h^\nu| \to 0$ as $\nu \to \infty$, and corresponding $p_{h^\nu} \in P_h^{n,m}$, with $\|p_{h^\nu}\|_n = 1$, such that p_{h^ν} satisfies the homogeneous equations corresponding to (2.3) and (2.5). By Lemma 2.1 there is a subsequence $\{p_{h^\nu}\}_{\nu=1}^\infty$ and a function $p \in C^{n-1}[0,1]$, such that $\|p_{h^\nu}-p\|_{n-1} \to 0$ as $\nu \to \infty$. Using the collocation equation we will show in part (iii) that in fact $p \in C^n[0,1]$, and that p is a nontrivial solution of the homogeneous problem (2.2a, b). This will contradict the first assumption of the theorem.

(ii) Assuming for the moment the existence of a unique p_h for all sufficiently small $|h|$, it remains to be shown that the bound (2.6) holds. If this is not the case, then again we can find a sequence of meshes $\{h^\nu\}_{\nu=1}^\infty$, with $|h^\nu| \to 0$ as $\nu \to \infty$, and for each mesh quantities $\{f^\nu(z_{j,i})\}^-$ and $\{\beta_l^\nu\}$ with $\max |f^\nu(z_{j,i})| \to 0$ and $\max_l |\beta_l^\nu| \to 0$ as $\nu \to \infty$, such that the corresponding unique solution $p_{h^\nu} \in P_{h^\nu}^{n,m}$ of (2.3), (2.5) has $\|p_{h^\nu}\|_n = 1$. Thus, as in part (i), there is a subsequence and a nontrivial $p \in C^{n-1}[0,1]$ such that $\|p_{h^\nu}-p\|_{n-1} \to 0$ as $\nu \to \infty$. Again we claim that p satisfies the homogeneous equations (2.2a, b).

(iii) Since the completion of the proof of (i) is very similar to the completion of the proof of part (ii), we only give the details of the latter.

Let $s \in [0,1]$, and consider the subsequence $\{p_{h^\nu}\}$. For each ν let j_ν be such that $s \in [x_{j_\nu}, x_{j_\nu+n}]$. Since $p_{j_\nu}^{(n)} \in P_{m-1}$ we can write

$$p_{j_\nu}^{(n)}(x) = \sum_{i=1}^m \psi_{j_\nu,i}(x) p_{j_\nu}^{(n)}(z_{j_\nu,i}),$$

where for each j_ν the functions $\{\psi_{j_\nu,i}(x)\}_{i=1}^m$ denote the Lagrange interpolating coefficients for the points $\{z_{j_\nu,i}\}_{i=1}^m$. Write $\psi_i \equiv \psi_{j_\nu,i}$ and $z_i^\nu \equiv z_{j_\nu,i}$. Then

$$(2.7)\quad \left|p_{j_\nu}^{(n)}(s)+\sum_{l=0}^{n-1} a_l(s)p^{(l)}(s)\right|$$

$$= \left|\sum_{i=1}^{m} \psi_i(s)p_{j_\nu}^{(n)}(z_i^\nu)+\sum_{l=0}^{n-1} a_l(s)p^{(l)}(s)\right|$$

$$= \left|\sum_{i=1}^{m} \psi_i(s)\left\{f^\nu(z_i^\nu)-\sum_{l=0}^{n-1} a_l(z_i^\nu)p_{j_\nu}^{(l)}(z_i^\nu)\right\}+\sum_{l=0}^{n-1} a_l(s)p^{(l)}(s)\right|$$

$$\leqq \left|\sum_{i=1}^{m} \psi_i(s)f^\nu(z_i^\nu)\right|+\sum_{l=0}^{n-1}\left|a_l(s)p^{(l)}(s)-\sum_{i=1}^{m} \psi_i(s)a_l(z_i^\nu)p_{j_\nu}^{(l)}(z_i^\nu)\right|$$

$$= \left|\sum_{i=1}^{m} \psi_i(s)f^\nu(z_i^\nu)\right|+\sum_{l=0}^{n-1}\left|\sum_{i=1}^{m} \psi_i(s)[a_l(s)p^{(l)}(s)-a_l(z_i^\nu)p_{j_\nu}^{(l)}(z_i^\nu)]\right|$$

$$\leqq mK_1 \max_{i,j}|f^\nu(z_{j,i})|+K_1\sum_{l=0}^{n-1}\sum_{i=1}^{m}\{|a_l(s)p^{(l)}(s)-a_l(z_i^\nu)p_{j_\nu}^{(l)}(z_i^\nu)|\}\ ,$$

where $K_1 \equiv \max_{i,j} \max_{[x_j, x_{j+n}]} |\psi_{j,i}(x)|$ can be chosen independent of the mesh, in view of the restriction on the location of the collocation points. This final expression becomes arbitrarily small as $\nu \to \infty$. It follows that $p_{h^\nu}^{(n)}$ converges to $[-\sum a_l(s)p^{(l)}(s)] \in C[0,1]$. In fact with the aid of Lemma 2.1 this convergence is seen to be uniform in s. Similar to the procedure in the proof of the lemma, we can extract a singlevalued continuous function $\tilde{p}_{h^\nu}^{(n-1)}$ from each $p_{h^\nu}^{(n-1)}$ and we have

$$\int_0^x \tilde{p}_{h^\nu}^{(n)}+\int_0^x \sum_{l=0}^{n-1} a_l p^{(l)} \to 0 \text{ as } \nu \to \infty\ .$$

(Note however that $\tilde{p}_{h^\nu}^{(n)}$ need not be continuous). Upon integration, using the continuity of $\tilde{p}_{h^\nu}^{(n-1)}$, it follows that

$$\tilde{p}_{h^\nu}^{(n-1)}(x)-\tilde{p}_{h^\nu}^{(n-1)}(0)+\int_0^x \sum_{l=0}^{n-1} a_l p^{(l)} \to 0 \text{ as } \nu \to \infty\ .$$

Taking the limit we obtain

$$p^{(n-1)}(x)-p^{(n-1)}(0)+\int_0^x \sum_{l=0}^{n-1} a_l p^{(l)} = 0\ .$$

This implies in particular that $p \in C^n[0,1]$. Differentiation gives

$$(2.8)\qquad p^{(n)}(x)+\sum_{l=0}^{n-1} a_l(x)p^l(x) = 0\ ,$$

i.e. $Lp(x)=0$. It is clear that p also satisfies the homogeneous boundary conditions (2.2b). Further with the aid of (2.7) and (2.8) we have

$$\|p_{h^\nu} - p\|_n \to 0 \text{ as } \nu \to \infty .$$

Since $\|p_{h^\nu}\|_n = 1$ this implies that $p \neq 0$. Hence a contradiction has been arrived at. ∎

REMARK. For any reasonable approximation space $P_h^{n,m}$, (i.e. a space whose elements satisfy appropriate matching or continuity conditions), it is easy to construct a sequence $p_{h^\nu} \to p \in C^{n-1}[0,1]$, as in Lemma 2.1. The difficulty lies in showing that $p \in C^n[0,1]$. In Theorem 2.2 this was accomplished quite easily, due to the fact that the nth derivative of any given polynomial component p_j is completely determined by the values of the nth derivative at the collocation points $z_{j,i}$. This local dependence is what characterizes "compact" difference approximations. (These can also be defined as difference methods that involve the unknown u_j at exactly $n+1$ consecutive mesh points.) It is also characteristic of collocation procedures with C^{n-1} piecewise polynomials for solving nth order differential equations. It is precisely for this type of approximation that a complete stability analysis on nonuniform meshes is relatively simple.

The theorem above does not impose any restrictions on the mesh. There is a condition that involves the location of the collocation points $z_{j,i}$. These points are assumed to be *locally* semi-uniform. Even this restriction can be removed by requiring some additional continuity. More continuity also allows bounding derivatives of order greater than n in the approximate solution. For proving these statements we need the following elementary fact concerning polynomial interpolation.

LEMMA 2.3. *Let* $s \in [0,1]$ *and let* $\{[a_\nu, b_\nu]\}_{\nu=1}^{\infty}$ *be a sequence of intervals in* $[0,1]$ *with* $s \in [a_\nu, b_\nu]$ *and* $\lim_{\nu\to\infty} (b_\nu - a_\nu) = 0$. *For each* ν *let* $\{z_i^\nu\}_{i=1}^{\mu}$ *be distinct points in* $[a_\nu, b_\nu]$. *Suppose* $g \in [0,1]$ *and* $g_\nu \in C^{\mu-1}[0,1]$ *with* $\max_{[a_\nu, b_\nu]} |g(x) - g_\nu(x)| \to 0$ *as* $\nu \to \infty$. *Also assume that* $\max_{[a_\nu, b_\nu]} |g_\nu^{(l)}(x)| \leqq c$, $0 \leqq l \leqq \mu - 1$. *Let* $q_\nu \in P_{\mu-1}$ *be the Lagrange interpolating polynomial of each* g_ν *on the points* $\{z_i^\nu\}_{i=1}^{\mu}$.

Then $\max_{[a_\nu, b_\nu]} |q_\nu(x) - g(x)| \to 0$ *as* $\nu \to \infty$.

PROOF. For each ν there is a point $\eta_\nu \in [a_\nu, b_\nu]$ such that $q_\nu^{(\mu-1)}(x) \equiv q_\nu^{(\mu-1)}(\eta_\nu) = g_\nu^{(\mu-1)}(\eta_\nu)$.

Thus
$$\begin{aligned}
&\max_{[a_\nu, b_\nu]} |q_\nu^{(\mu-1)}(x) - g_\nu^{(\mu-1)}(x)| \\
&= \max |q_\nu^{(\mu-1)}(x) - g_\nu^{(\mu-1)}(\eta_\nu) + g_\nu^{(\mu-1)}(\eta_\nu) - g_\nu^{(\mu-1)}(x)| \\
&= \max |g_\nu^{(\mu-1)}(\eta_\nu) - g_\nu^{(\mu-1)}(x)| \leq 2c .
\end{aligned}$$

It follows that $\max |q_\nu(x) - g_\nu(x)| \leqq 2c(b_\nu - a_\nu)^{\mu-1}$ and that

$$\max|q_\nu(x)-g(x)| \leqq \max|q_\nu(x)-g_\nu(x)|+\max|g_\nu(x)-g(x)|$$
$$\leqq 2c(b_\nu-a_\nu)^{\mu-1}+\max|g_\nu(x)-g(x)| . \quad \blacksquare$$

THEOREM 2.4. *Let* (2.2a, b) *have the zero solution only. For each j let the collocation points $\{z_{j,i}\}_{i=1}^m$ be distinct and contained in $[x_j, x_{j+n}]$. Assume that f, $a_l \in C^{m-1}[0,1]$. Then there exist positive constants K and δ, such that* (2.3), (2.5) *admit a unique solution $p_h \in P_h^{n,m}$ and such that*

$$\|p_h\|_{n+m-1} \leqq K\{\max_{0\leqq k\leqq m-1} \|f^{(k)}\|_\infty + \max_{1\leqq l\leqq n} |\beta_l|\} ,$$

whenever $|h| \in (0,\delta]$.

PROOF. Again the existence should be established first. As in Theorem 2.2 the proof of the existence part closely follows the proof of the second conclusion of the theorem. Hence only the latter is given below.

If the bound does not hold, then one can find sequences $f^\nu \in C^{m-1}[0,1]$, β_l^ν; with $\|f^{\nu(l)}\|_\infty \to 0$, $0\leqq l\leqq m-1$, and $|\beta_l^\nu| \to 0$, $1\leqq l\leqq n$, as $\nu \to \infty$; and corresponding solutions $p_{h^\nu} \in P_{h^\nu}^{n,m}$ with $\|p_{h^\nu}\|_{n+m-1}=1$. By Lemma 2.1 there is a subsequence $\{p_{h^\nu}\}_{\nu=1}^\infty$ and a function $p \in C^{n-1}[0,1]$ such that $\|p_{h^\nu}-p\|_{n-1} \to 0$ as $\nu \to \infty$.

Now reconsider the estimate (2.7):

$$\left|p_{j_\nu}{}^{(n)}(s)+\sum_{l=0}^{n-1} a_l(s)p^{(l)}(s)\right| \leqq \left|\sum_{i=1}^{m} \psi_i(s)f^\nu(z_i^\nu)\right| +$$
$$+\sum_{l=0}^{n-1} \left|a_l(s)p^{(l)}(s)-\sum_{i=1}^{m} \psi_i(s)a_l(z_i^\nu)p_{j_\nu}^{(l)}(z_i^\nu)\right| .$$

Note that Lemma 2.3, (with $\mu=m$), applies to each of the $n+1$ expressions that appear inside absolute value signs above. Therefore the righthand side of the inequality can be made arbitrarily small by choosing ν sufficiently large. Again this convergence can be shown to be uniform in s. (The details require use of the final inequality in the proof of Lemma 2.3). By the same argument given in the proof of Theorem 2.2 it now follows that $p \in C^n[0,1]$ and that $Lp=0$. Also p satisfies the homogeneous boundary conditions (2.2b). However we cannot conclude at this point that $p \not\equiv 0$, unless $m=1$, since for this we must show first that $\|p_{h^\nu}-p\|_{n+m-1} \to 0$ as $\nu \to \infty$. Assume therefore that $m>1$. Rewrite the equation $Lp=0$ as

$$p^{(n)}(x) = -\sum_{l=0}^{n-1} a_l(x)p^{(l)}(x) .$$

Thus $p^{(n)} \in C^1[0,1]$, i.e. $p \in C^{n+1}[0,1]$, and

$$L^{[1]}p \equiv (Lp)' \equiv p^{(n+1)}(x) + \sum_{l=0}^{n} a_l^{[1]}(x)p^{(l)}(x) = 0 .$$

In view of the continuity assumptions on the coefficient functions $a_l(x)$ we can repeat this argument another $m-2$ times and finally conclude that $p \in C^{n+m-1}[0,1]$.

By Rolle's theorem there are points $\xi_i \equiv \xi_{j_\nu,i}^{[1]}$ in $(x_{j_\nu}, x_{j_\nu+n})$ such that $L^{[1]}p_{j_\nu}(\xi_i) = f^{\nu'}(\xi_i)$, $1 \leqq i \leqq m-1$. Let $\psi_i \equiv \psi_{j_\nu,i}^{[1]}$ be the Lagrange interpolating coefficients for the points ξ_i. Then

$$\left|p_{j_\nu}^{(n+1)}(s) + \sum_{l=0}^{n} a_l^{[1]}(s)p^{(l)}(s)\right| \leqq \left|\sum_{i=1}^{m-1} \psi_i(s) f^{\nu'}(\xi_i)\right| +$$

$$+ \sum_{l=0}^{n} \left|a_l^{[1]}(s)p^{(l)}(s) - \sum_{i=1}^{m-1} \psi_i(s) a_l^{[1]}(\xi_i) p_{j_\nu}^{(l)}(\xi_i)\right| .$$

With the aid of Lemma 2.3 it follows that the right hand side of the inequality above goes to zero uniformly in s as $\nu \to \infty$. Thus $p_{j_\nu}^{(n+1)} \to p^{(n+1)}$. The above can be repeated another $m-2$ times to finally yield

$$\|p_{h^\nu} - p\|_{n+m-1} \to 0 \text{ as } \nu \to \infty .$$

Since $\|p_{h^\nu}\|_{n+m-1} = 1$ we can now conclude that $p \not\equiv 0$, which is a contradiction. ∎

To derive some error estimates the well-known concept of truncation error is required. Let $\varrho_h \equiv \{\varrho_j\}_{j=0}^{J-n}$, where $\varrho_j \in P_{n+m-1}$ interpolates $y(x)$ at $\{x_{j+i}\}_{i=0}^{n} \cup \{t_{j,i}\}_{i=1}^{m-1}$. Here we have introduced points $t_{j,i} \in [x_j, x_{j+n}]$. (Coinciding interpolation points denote Hermite interpolation.) Thus

$$y(x) - \varrho_j(x) = r_j(x) d_j(x), \text{ where } r_j(x) \equiv \prod_{i=0}^{n} (x - x_{j+i}) \prod_{i=1}^{m-1} (x - t_{j,i})$$

and where $d_j(x)$ denotes the appropriate divided difference. Define $\tau_h \equiv \{\tau_j\}_{j=0}^{J-n}$, with $\tau_j(x) \equiv Lp_j(x) - L\varrho_j(x)$. The local truncation errors can now be defined as the values of $\tau_j(x)$ at $x = z_{j,i}$. We have

$$\tau_j(z_{j,i}) = f(z_{j,i}) - L\varrho_j(z_{j,i}) = L(y - \varrho_j)(z_{j,i}) = L(r_j d_j)(z_{j,i}) .$$

If $y(x)$ is sufficiently differentiable, then $\tau_j(z_{j,i}) = O(|h|^m)$ regardless of the choice of collocation points $z_{j,i}$. For certain special choices of the $z_{j,i}$ this estimate can be improved. Certain of these special points can be found by evaluating $L(r_j d_j)(z_{j,i})$ and equating the leading terms to zero. Examples of this procedure are given in [1].

THEOREM 2.5. *Let the homogeneous problem* (2.2a, b) *only admit the zero solution. Let* $a_l \in C[0,1]$, $y \in C^{n+m}[0,1]$ *and assume that* $z_{j,i} \in [x_j, x_{j+n}]$, *with* $\min_{i_1 \neq i_2} |z_{j,i_1} - z_{j,i_2}| \geqq c_1 |x_{j+n} - x_j|$. *Also assume that* $\max_{i,j} |\tau_j(z_{j,i})| \leqq c_2 |h|^{m+\sigma}$ *for some nonnegative integer* σ, *and in addition* $|B_l \varrho_h - \beta_l| \leqq c_3 |h|^{m+\mu}$, $1 \leqq k \leqq n$, $\mu \geqq \sigma$. *Here* c_1,

c_2 and c_3 are constants that do not depend on j and h, when $|h|$ is small enough. Then there exist positive constants C and δ such that

$$\max_j \max_{[x_j, x_{j+n}]} |p_j^{(l)}(x) - y^{(l)}(x)| \leqq C|h|^{\min\{m+\sigma, n+m-l\}}, \quad 0 \leqq l \leqq n,$$

whenever $|h| \in (0, \delta]$.

PROOF.

$$|y^{(l)}(x) - p_j^{(l)}(x)| \leqq |y^{(l)}(x) - \varrho_j^{(l)}(x)| + |p_j^{(l)}(x) - \varrho_j^{(l)}(x)| .$$

The first term on the right hand side of this inequality is a local interpolation error. The second term can be estimated with Theorem 2.2. We have

$$\max_j \max_{[x_j, x_{j+n}]} |y^{(l)}(x) - p_j^{(l)}(x)|$$

$$\leqq c_4|h|^{n+m-l} + K\left\{\max_{i,j} |\tau_{j,i}(z_{j,i})| + \max_l |B_l\varrho_h - \beta_l|\right\}$$

$$\leqq c_4|h|^{n+m-l} + K(c_2 + c_3)|h|^{m+\sigma} \qquad \blacksquare$$

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