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A Numerical Technique for Bifurcation Problems in Delay Differential Equations

by

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1. Introduction In a previous contribution to this Conference Series [EJD2] a continuation technique for branches of periodic solutions to autonomous systems of ordinary differential equations was presented. Since then the corresponding software has been considerably generalized and applied to a variety of problems, including some nonlinear parabolic partial differential equations. Rather interesting bifurcation phenomena were discovered in many models. See for example [EJD/RFH] for results on the exothermic $A \to B \to C$ reaction in a continuously stirred medium.

Another class of problems, where even simple models may exhibit very complex global behaviour, is that of functional differential equations [JH1]. These equations arise in various areas of applications, notably in population dynamics [NM] and in physiological models [M/G]. An important subclass is that of differential equations having a time delay. A major difference between delay equations and autonomous systems without delays is the dimensionality of the problem. For example, the determination of a periodic solution to an autonomous system without delays requires an initial vector to be found, whereas an initial function is required for finding a periodic solution to the delay equation. Thus the periodicity problem passes from a finite dimensional to an infinite dimensional problem.

In addition to stationary and periodic solutions to delay differential equations, one is often also interested in the existence or non-existence of a-periodic solutions such as solutions on a torus or chaotic solutions. Such behaviour is observed in physical reality and equally well in various mathematical models. See for example [M/G]. Numerical simulation can be done using initial value methods that allow for a time delay. This approach is very straightforward and generally provides good insight in the actual stable behaviour of the system. In particular the determination of stable a-periodic solutions would appear to be possible only by this approach. (Certainly this statement holds for chaotic solutions, but not necessarily for solutions on a torus.)

Initial value techniques cannot be used directly for computing solutions that are not asymptotically stable. If a delay problem contains a free parameter, and if the object of study is the dependence of the solution structure on that parameter, then unstable solutions may have to be computed simply to reach stable and hence physical solutions elsewhere. Further, a branch (curve) in solution-parameter space may have limit points (turning points), near which the application of initial value techniques is difficult if not impossible.

In this paper we describe a general procedure for treating the Hopf bifurcation problem in a scalar single delay differential equation containing a free parameter, and for computing the associated branch of periodic solutions. The stability properties of the branch do not affect the computational procedure and limit points present no difficulty. Furthermore, most of the techniques generalize directly to higher order systems with multiple delays. The main ideas are those of [HBK],[EJD2],[ADJ],[D/J/K], however certain modifications are required for taking the delay into account. The basic approximation technique consists of truncated Fourier series and collocation. As the numerical examples in Section 3 indicate, this procedure is very accurate, as expected, at least near the bifurcation point.

Little previous numerical work has been reported on the Hopf bifurcation problem for delay differential equations. The main reference appears to be [KPH], where a different approach is considered. Our method has the advantage of having an automatic procedure for detecting Hopf bifurcation and for starting the associated branch of periodic solutions. Also, our computational technique for periodic solutions applies without change at or near limit points and requires no essential change at the Hopf bifurcation point.

2. <u>The Hopf Bifurcation Problem</u> We consider the differential equation

(2.1) $$u'(t) = f(u(t), u(t-\tau), \lambda)$$

where τ is a fixed delay and λ a free parameter. It is possible to identify λ with τ. (But see [JH2].) Stationary (steady state) solutions satisfy $g(u,\lambda) \equiv f(u,u,\lambda) = 0$, and the steady state bifurcation structure can therefore be computed numerically by the corresponding techniques discussed in [HBK], [EJD2]. In fact we have transplanted the steady state section of our computer program AUTO for autonomous ODE's without delays into our current computer code DLAY.

The detection of Hopf bifurcation points on a branch of stationary solutions proceeds differently however. Necessary and sufficient conditions for a solution point $(u_\emptyset, \lambda_\emptyset)$ to be a Hopf bifurcation point are well known [JH1],[JH2]. The basic necessary condition can be formally derived as follows : We seek a solution of the form

$$(2.2) \qquad u(t) = e^{iyt}, \quad y \equiv \frac{2\Pi}{\rho}.$$

to the linearization of (2.1), i.e. to the linear delay differential equation

$$(2.3) \qquad u'(t) = A(\lambda)\ u(t) + B(\lambda)\ u(t-\tau),$$

with $A(\lambda) = \frac{\partial f}{\partial u}(u_\emptyset, u_\emptyset, \lambda)$, $B(\lambda) = \frac{\partial f}{\partial v}(u_\emptyset, u_\emptyset, \lambda)$, $f \equiv f(u,v,\lambda)$.

Note that if e^{iyt} is a solution of (2.3) then so is e^{-iyt} and hence sin(yt) and cos(yt) also. Substitution of (2.2) into (2.3) results in the characteristic equation

$$(2.4) \qquad iy - A(\lambda) - e^{-iy\tau} B(\lambda) = \emptyset,$$

where we seek a real solution y. More accurately, along a steady state solution branch $(u(s), \lambda(s))$ we want to find those solution points $(u_\emptyset, \lambda_\emptyset)$ for which (2.4) has a real solution y. Equivalently we want to determine λ for which a solution z of

$$(2.5) \qquad z - A(\lambda) - e^{-\tau z} B(\lambda) = \emptyset,$$

crosses the imaginary axis. For systems of delay differential equations, (2.5) amounts to a rather difficult nonlinear eigenvalue problem. However, in the one dimensional case the necesssary condition for Hopf bifurcation is easily derived explicitly. Separation of real and complex part of (2.4) gives the equations

$$(2.6a) \qquad y + \sin(\tau y)\ B(\lambda) = \emptyset,$$

$$(2.6b) \qquad A(\lambda) + \cos(\tau y)\ B(\lambda) = \emptyset.$$

Elimation of y from (2.6) results in the single equation

$$(2.7) \qquad \sigma(\lambda) \equiv A(\lambda) + B(\lambda) \cos(\ \tau\ [B(\lambda)^2 - A(\lambda)^2]^{1/2}) = \emptyset.$$

The value of the expression in (2.7) is monitored by the program while the steady state bifurcation picture is determined and stationary solutions where the value vanishes are located accurately. These points are potential Hopf bifurcation points. If for such a candidate point the equations (2.6a,b) do not specify y, then the point is rejected as a bifurcation point. If

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on the other hand (2.6a) and (2.6b) do specify y, then we have located a Hopf bifurcation point, provided only that the root in question of (2.5) crosses the imaginary axis at a non-vertical angle. (See [JH1].) However, the latter sufficient condition is easily shown to be equivalent to $\sigma'(\lambda)$ being nonzero at the root. This is easy to verify numerically. Moreover it implies that $\sigma(\lambda)$ changes sign at the root, which is helpful in the automatic detection of the bifurcation.

We now turn to the actual computation of branches of periodic solutions associated with the Hopf bifurcation point. First we describe the procedure for continuing a branch away from the Hopf bifurcation. After this we show how essentially the same procedure can be used to start the computation of a branch at the Hopf bifurcation points.

For reasons of differentiability, in view of the presence of the delay, it is convenient to seek approximate solutions in the form of a trigonometric expansion. In addition this choice forces periodicity, which for delay problems cannot be obtained by simply requiring $u(0) = u(\rho)$. Also it is convenient to scale the independent variable by the factor $\frac{2\Pi}{\rho}$. (Note that ρ is to be determined also.) This transforms the equation (2.1) into

$$(2.8) \qquad u'(t) = \frac{\rho}{2\Pi} f(u(t)\ ,\ u(t - \frac{2\Pi}{\rho}\tau)\ ,\ \lambda),$$

to which we now want to determine 2Π periodic solutions. Discretization occurs via collocation. More precisely, we seek

$$(2.9a) \qquad u_n(t) = \sum_{k=-n}^{n} c_k e^{ikt} = a_0 + \sum_{k=1}^{n} a_k \sin(kt) + \sum_{k=1}^{n} b_k \cos(kt),$$

that satisfies the differential equation (2.8) at 2n+1 equally spaced points $t_j = j\Delta t$, $\Delta t \equiv \frac{2\Pi}{2n+1}$, i.e.,

$$(2.9b) \qquad u_n'(t_j) = \frac{\rho}{2\Pi} f(u_n(t_j),\ u_n(t_j - \frac{2\Pi}{\rho}\tau),\lambda),\ j=0,1,\ \dots\ ,\ 2n.$$

Assume we have computed ℓ solution points $(u_n^{(k)}(t),\rho^{(k)},\lambda^{(k)})$, $k=0,1,\dots,\ell-1$, along a given branch of periodic solutions. To remove the inherent nonuniqueness of the next solution point $(u^{(\ell)}(t),\rho^{(\ell)},\lambda^{(\ell)})$ (due to the freedom of translation in time) we impose the anchor equation

$$(2.10) \qquad \int_0^{2\Pi} (u_n^{(\ell)}(t) - u_n^{(\ell-1)}(t))\ u_n^{(\ell)'}(t)\ dt = 0.$$

The rationale behind this choice is described in [EJD2] for autonomous system without delays, and it applies equally well here. (See also below.) To fully specify $(u_n^{(\ell)},\rho^{(\ell)},\lambda^{(\ell)})$, we require the "distance" between $(u_n^{(\ell)},\rho^{(\ell)},\lambda^{(\ell)})$ and $(u_n^{(\ell-1)},\rho^{(\ell-1)},\lambda^{(\ell-1)})$ to equal a prespecified increment Δs,

$$(2.11)\quad \int_{\emptyset}^{2\Pi} (u_n^{(\ell)}(t)-u_n^{(\ell-1)}(t))^2\,dt + (\rho^{(\ell)}-\rho^{(\ell-1)})^2 + $$

$$+ (\lambda^{(\ell)}-\lambda^{(\ell-1)})^2 = \Delta s^2.$$

Note that to allow the computation to proceed past limit points, we also treat $\lambda^{(\ell)}$ as one of the unknowns as is done in [HBK]. This in fact necessitates the addition of (2.11). In view of the form of the approximate solution the equations (2.9b), (2.1Ø) and (2.11) can be expressed in terms of the Fourier coefficients a_k and b_k. Thus the system of 2n+3 nonlinear algebraic equations that must be solved for every step along the branch consists of

$$(2.12)\quad \sum_1^n k(a_k\cos(kt_j)-b_k\sin(kt_j))=$$

$$\frac{\rho}{2\Pi}f(a_\emptyset + \sum_1^n(a_k\sin(kt_j) + b_k\cos(kt_j)),$$

$$a_\emptyset + \sum_1^n(a_k\sin(k(t_j-\frac{2\Pi}{\rho}\tau)) + b_k\cos(k(t_j-\frac{2\Pi}{\rho}\tau))),\lambda),$$

$j=\emptyset,1,\dots,2n.$

$$(2.13)\quad \sum_{k=1}^n k\,(a_k^{(\ell-1)}b_k-a_kb_k^{(\ell-1)}) = \emptyset,$$

and

$$(2.14)\quad 2\Pi\,(a_\emptyset-a_\emptyset^{(\ell-1)})^2 + \Pi\sum_{k=1}^n[(a_k-a_k^{(\ell-1)})^2 +(b_k-b_k^{(\ell-1)})^2] +$$

$$+ (\rho-\rho^{(\ell-1)})^2 + (\lambda-\lambda^{(\ell-1)})^2 = \Delta s^2.$$

Above we have omitted all superscripts (ℓ). To solve (2.12), (2.13) and (2.14) for the a_k, b_k, ρ and λ, we use a Newton-Chord method. Note that the equations actually have two solutions viz. $(u^{(\ell-2)},\rho^{(\ell-2)},\lambda^{(\ell-2)})$ and $(u^{(\ell)},\rho^{(\ell)},\lambda^{(\ell)})$, assuming that Δs is held fixed. This does not normally cause difficulties, as an accurate initial approximation to $(u^{(\ell)},\rho^{(\ell)},\lambda^{(\ell)})$ can be obtained by extrapolation from the two preceding solution points on the branch. Moreover, if needed, this difficulty is easily removed by a minor modification of (2.14). (See [HBK].)

We now consider the starting procedure for a branch of periodic solutions. Thus a Hopf bifurcation point $(u^{(\emptyset)}, \rho^{(\emptyset)}, \lambda^{(\emptyset)})$ is given, with $u^{(\emptyset)}$ a stationary solution, i.e. a constant. The problem consists of finding a first solution point $(u^{(1)}, \rho^{(1)}, \lambda^{(1)})$, with non-stationary u, on the branch. Equations (2.12) and (2.14) apply without change. In particular (2.14) regulates the distance of the first point $(u^{(1)}, \rho^{(1)}, \lambda^{(1)})$ to the Hopf bifurcation point. Note that $a_\emptyset = u^{(\emptyset)}$, $a_k = b_k = \emptyset$, $k \geq 1$. However the anchor equation (2.13) requires modification, as the coefficients of all a_k and b_k are zero, which would result in a singular Jacobian. To derive a suitable modification we recall the derivation of the anchor equation (2.1∅) or (2.13) (cf. [EJD2]). The underlying idea is to "align" $u^{(\ell)}$ and $u^{(\ell-1)}$ using the freedom we have to translate the new solution $u^{(\ell)}$. Thus we minimize

$$\int_\emptyset^{2\Pi} (u^{(\ell)}(t+\sigma) - u^{(\ell-1)}(t))^2 \, dt$$

over sigma. This yields (2.1∅). But when $\ell=1$, i.e. at the Hopf bifurcation point, $u^{(\emptyset)}$ is constant and the above integral is independent of σ. However, we have at our disposal an asymptotic estimate for the periodic solution in a neighbourhood of the bifurcation point. This estimate follows from (2.2),(2.3) and, recalling our scaling of the independent variable, it takes the form

$$(2.15) \quad u^{(\varepsilon)}(t) = u^{(\emptyset)} + \varepsilon \, [\, c_1 \cos(t) + c_2 \sin(t) \,] + O(\varepsilon^2).$$

The freedom of phase shift is still present in (2.15) and allows omitting, say, the sine term. Thus near the bifurcation point the periodic solution can be represented asymptotically by

$$(2.16a) \qquad u^{(\varepsilon)}(t) = u^{(\emptyset)} + \varepsilon \cos(t) + O(\varepsilon^2).$$

Further, for compatibility with (2.11) we take
$\varepsilon = \Delta s \,/ \int_\emptyset^{2\Pi} \cos^2(t)\, dt = \Delta s/\Pi$. It is also known (and easy to show) that

$$(2.16b) \qquad \lambda^{(\varepsilon)} = \lambda^{(\emptyset)} + O(\varepsilon^2),$$

and

$$(2.16c) \qquad \rho^{(\varepsilon)} = \rho^{(\emptyset)} + O(\varepsilon^2).$$

It is natural now, given a particular representation the periodic solution near the bifurcation point, to "align" $u^{(1)}$ in the first solution point $(u^{(1)},\rho^{(1)},\lambda^{(1)})$ with the asymptotic estimate $\overline{u}^{(1)} = u^{(0)} + \Delta s \cos(t) / \Pi$ of $u^{(1)}$. Thus instead of (2.10) we use the anchor equation

$$(2.17) \quad \int_0^{2\Pi} (u_n^{(1)}(t) - \overline{u}_n^{(1)}(t))\, \overline{u}_n^{(1)\prime}(t)\, dt = 0.$$

In terms of the Fourier coefficients this becomes

$$(2.18) \qquad a_1 = 0.$$

Of course this simple equation could have been arrived at more directly. However the derivation above applies to more general bifurcation problems. The starting procedure now consists of solving (2.12),(2.14) and (2.18). The initial estimate in Newton's method to the first solution point $(u^{(1)},\rho^{(1)},\lambda^{(1)})$ on the branch can also be obtained from (2.16). It should be noted that our starting procedure is only slightly different from the continuation procedure elsewhere on the branch, thus keeping down the overall programming effort.

If a branch of periodic solutions or a portion thereof consists of asymptotically stable solutions and if these are not near limit points or bifurcation points, then it would be much more efficient to use an initial value problem solver to compute these stable solutions. Thus our solver could be complemented by initial value techniques. For systems of delay equations with large dimension this might indeed be necessary. However, we have not taken this approach, i.e. the procedure described in this section is applied equally well if the solutions are asymptotically stable. One reason for this is that it allows the period ρ to be computed very accurately. A second, more important reason, is that it makes the detection and accurate location of secondary periodic bifurcations easy. For example, to detect an ordinary (i.e., no period doubling or a-periodicity) secondary periodic bifurcation we need only monitor the determinant of the Jacobian corresponding to the equations (2.12), (2.13), (2.14). Period doubling bifurcations can be detected in a similarly easy way.

3. Numerical Examples

Example 3.1 In order to demonstrate the accuracy of the method and to check the correctness of our implementation, we have recomputed the example given in [KPH]. The equation is

$$u'(t) = -\lambda\, u(t-1)\, \frac{1+u(t-1)^2}{1+u(t-1)^4}.$$

For λ positive and along the zero stationary solution branch (1), our program signals candidate Hopf bifurcation points (roots of (2.7)) at $\lambda = \pi/2$, $4\pi/2$, $5\pi/2$, $8\pi/2$, $9\pi/2$, $12\pi/2$, ... Only the first of these and every second one following are determined to be actual bifurcation points ((2.6) solvable for y, and $\sigma'(\lambda)$ nonzero at the root). The bifurcation diagram is shown in Figure 1. In the diagram branch (1) represents the zero stationary solution, while branches (2),(3) and (4) are the primary branches of periodic solutions. A secondary periodic bifurcation occurs along branch (2) at λ=4.67. The bifurcating branch of periodic solutions has been labelled (5) in Figure 1. This secondary bifurcation is an ordinary bifurcation and it is detected by our program DLAY as the sign changes in the determinant of the Jacobian of Equations (2.12),(2.13),(2.14). With our formulation of the problem, the techniques of [HBK] can be used to switch from one branch to another at the secondary bifurcation. As is also noted in [KPH], such an ordinary bifurcation is not structurally stable. Indeed, for insufficiently accurate discretization the usual perturbed bifurcation is observed. (For a general discussion of the effect of discretization see [WJB],[WJB/EJD],[EJD1].) In this example the bifurcation diagram can be computed with n=12 or less in (2.9a). Near the secondary bifurcation we have used n=20.

The equation of this example has the property that along the primary periodic solution branches the period actually remains constant. This is rather unusual and allows us to compare the numerically obtained period to the actual period. For example along the first bifurcating branch of periodic solutions (2) the period remains equal to 4. In the table below we list the numerically observed period on this branch at λ=3.0 for various choices of n in (2.9a). As is evident the convergence is indeed very rapid. (Along branches (3) and (4) in Figure 1 the period also remains constant and equals 4/5 and 4/9 respectively.)

n	ρ
2	4.724
4	4.135
6	3.975
8	3.9987
10	3.9985
12	3.99984
16	4.000000

In this problem the zero stationary solution (1) is stable up to the first Hopf bifurcation point. Periodic branch (2) is stable between the limit point and the secondary periodic bifurcation. The upper portion of branch (5) (past the limit point) is also stable. All other solutions indicated in the bifurcation diagram are asymptotically unstable. The actual solution u(t) at points 31 and 51 on branches (3) and (5) respectively is shown in Figure 2. Thus solution 31 is unstable and 51 is stable. For both solution points λ approximately equals 8.0. To illustrate the transitional behaviour of this equation we have also solved the differential equation for $\lambda=8.0$ using a simple initial value problem solver. As initial data on the time interval $[-1,0]$ we have taken a sine function that approximates the unstable solution 31. The dynamic response of the differential equation to this starting condition is shown in Figure 3. Initially the solution oscillates near the unstable solution 31 which has period 0.8. Then a quick transition takes place from the unstable oscillation to the stable periodic solution 51 which has period approximately equal to 5.5. (The period does not remain constant along the secondary periodic branch (5).)

Example 3.2 As a second example we consider the fully delayed logistic equation

$$u'(t) = \lambda\, u(t-1)\ [\ 1 - u(t-1)\ ].$$

The bifurcation diagram (see Figure 4) has three stationary solution branches, viz. (1) $u \equiv 0$, (2) $\lambda = 0$ and (3) $u \equiv 1$. The stationary solution $u \equiv 1$ is stable only up to $\lambda=\pi/2$, where there is a Hopf bifurcation. The corresponding branch of periodic solutions (4) is stable. This branch becomes "vertical" with $\lambda \approx 2.478$, and along it the period becomes very large. Some solutions (again scaled to the interval $[0,2\pi]$) are shown in Figure 5. For λ greater than 2.36 the periodic solutions are no longer positive, signalling extinction if the differential equation is interpreted as a population model. The last nonnegative solution has label 42 in Figures 4 and 5. Its period equals 5.37. Solution points 41 and 43 have period 4.09 and 9.54 respectively.

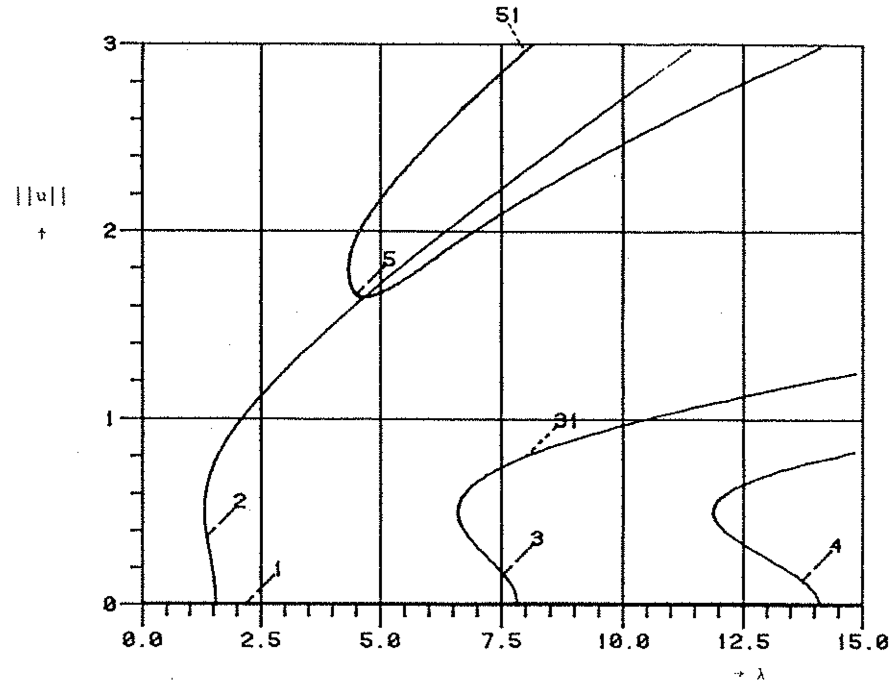

FIGURE 1

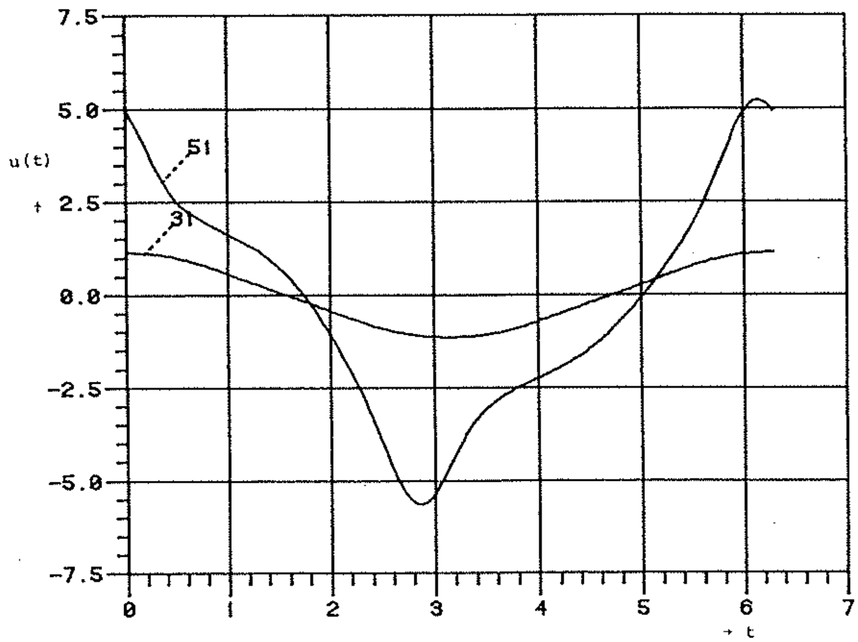

FIGURE 2

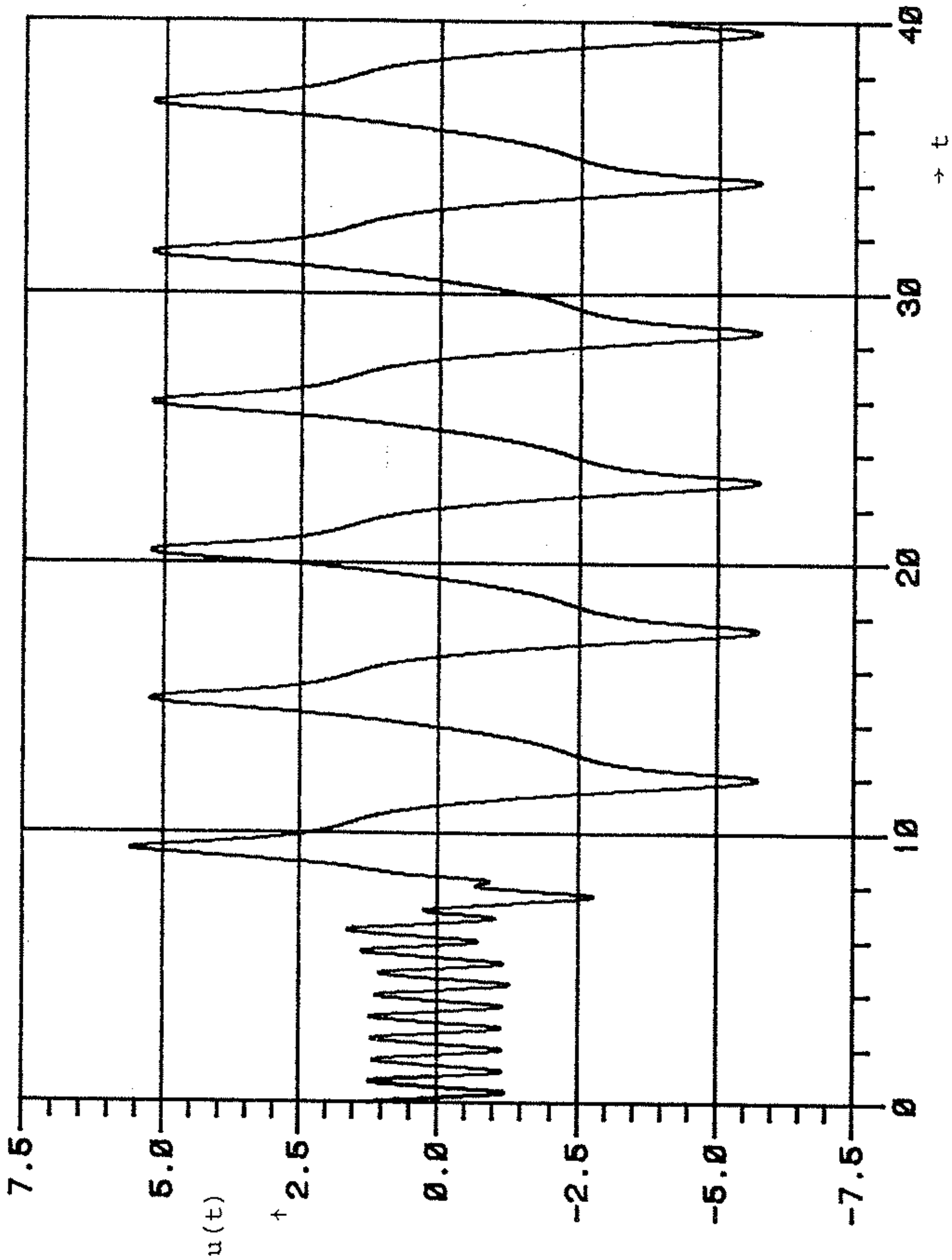

FIGURE 3

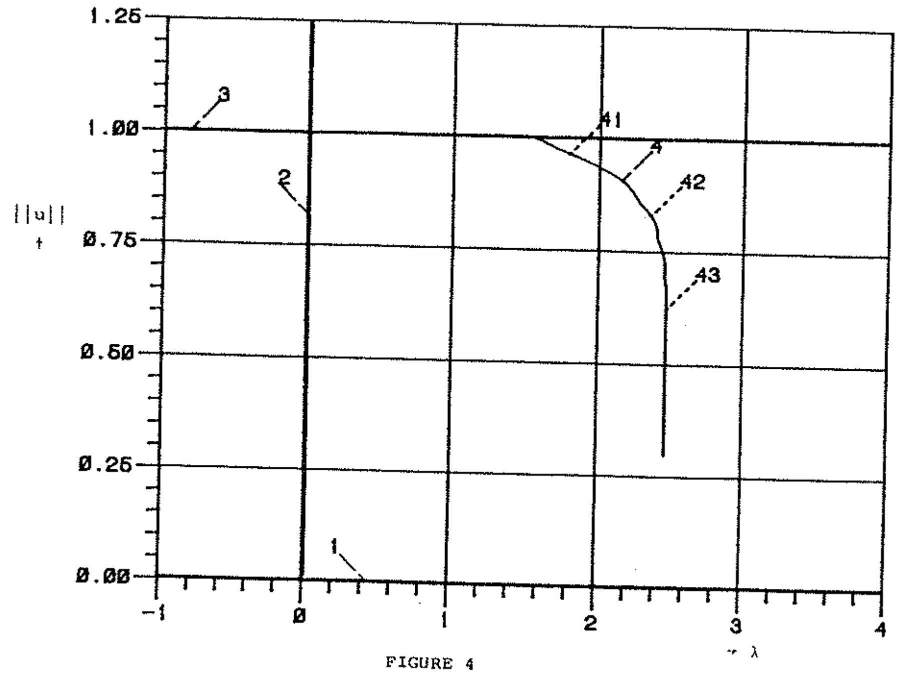

FIGURE 4

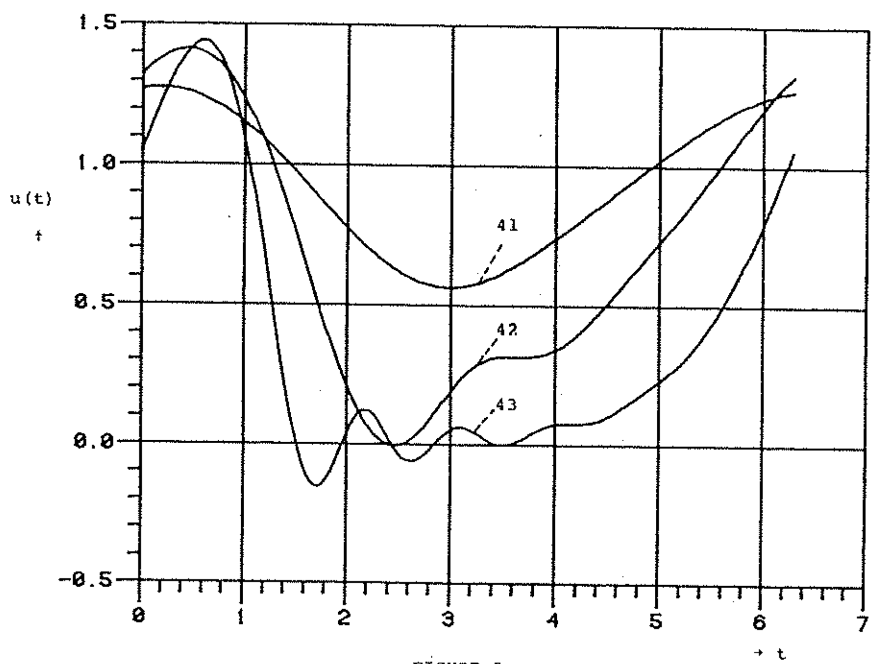

FIGURE 5

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